Macroscopic dynamics of globally coupled systems

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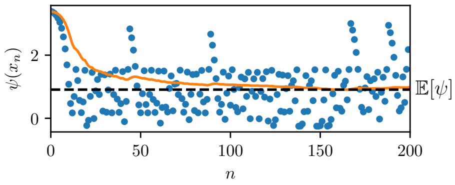
Chaotic systems

Statistics of chaotic systems

Things we are interested in:

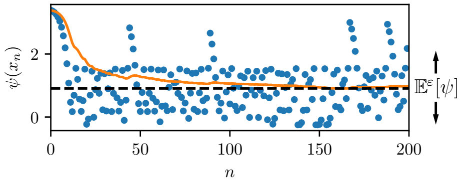
- Existence of chaos! (Positive Lyapunov exponents)
- Physical measures:

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi(T^n(x)) = \int \psi(x)\,\mathrm{d}\nu(x), \text{ Lebesgue a.e. } x$$



Statistics of chaotic systems

- Mixing rates, statistics such as large deviations
- Response of physical measures to dynamical perturbations (e.g. linear response)



Tractable chaotic systems

For rigorous results, some strong geometrical constraints on the dynamics are needed. Results in:

- $1 + \epsilon$ dimensions (e.g. logistic, Hénon)
- Systems with (some) hyperbolicity

Real chaotic systems

Consider the most (practically) important examples of chaotic systems:

- Statistical mechanics (incl. non-equilibrium)
- Turbulent fluid flow
- Global climate systems

They are theoretically intractable:

A. High-dimensional with many positive Lyapunov exponents
B. Non-hyperbolic.

Real chaotic systems

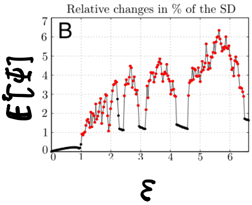
How to make sense of these systems?

Chaotic hypothesis (Gallavotti-Cohen '95)

The macroscopic dynamics of a (high-dimensional) chaotic system on its attractor can be regarded as a transitive hyperbolic ("Anosov") evolution.

Ergo: we expect all the same nice statistics as in hyperbolic systems.

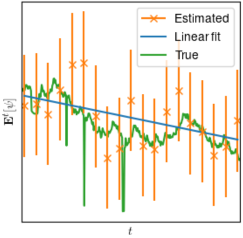
Real chaotic systems



Chekroun et al., 2014

However (examples from response theory):

- Sometimes one or more of these properties fail (e.g. Chekroun et al. '14)
- Maybe more failures are obscured by finite data effects (Gottwald, W. & Wouters '16)



Would like to study the (range of) validity of the chaotic hypothesis, rigorously. . .

Globally coupled systems

"Simple complex system": globally coupled systems of M subunits $x^{(j)}$ with

$$x_{n+1}^{(j)} = f\left(x_n^{(j)}; \frac{1}{M}\sum_{m=1}^{M}\phi(x_n^{(m)}, x_n^{(j)})\right), \; j = 1, \dots, M$$

$f(\cdot;\Phi)$ chaotic, ϕ a coupling function (Kaneko '88).
Example of these are attractively coupled systems (work of LS Young, Fernandez, Sélley, . . .):

$$x_{n+1}^{(j)} = f\left(x_n^{(j)} + \frac{K}{M}\sum_{m=1}^{M}(x_n^{(m)} - x_n^{(j)})\right)$$

Mean-field coupled systems

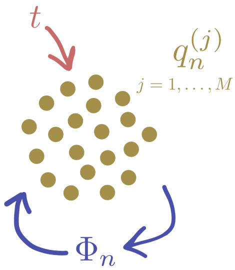
Subset of these: *mean-field* coupled systems where $\phi(x^{(m)}, x^{(j)}) \equiv \phi(x^{(m)})$. Write mean-field

$$\Phi_n = \frac{1}{M} \sum_{m=1}^{M} \phi(x_n^{(m)})$$

Then have dynamics

$$x_{n+1}^{(j)} = f(x_n^{(j)}, \Phi_n) =: f_{\Phi_n}(x_n^{(j)}),\ j = 1, \ldots, M$$

We will show these have interesting and problematic dynamics...



Thermodynamic limit reduction

The $x^{(j)}$'s are exchangeable. So we can formulate in terms of empirical measure of $x^{(j)}$s:

$$\mu_n = \frac{1}{M} \sum_{j=1}^{M} \delta_{x_n^{(j)}}$$

so that system becomes

$$\Phi_n = \int \phi \, \mathrm{d}\mu_n$$
$$\mu_{n+1} = f^*_{\Phi_n} \mu_n$$

This gives dynamical system in μ_n:

$$\mu_{n+1} = F(\mu_n) := f^*_{\int \phi \, \mathrm{d}\mu_n} \mu_n$$

Taking $M \to \infty$ we might expect μ_0 to converge to a continuous distribution.

Thermodynamic limit reduction

We can study measure dynamics using the linear transfer operator $\mathcal{L}_f$:

$$\mathcal{L}_f h \, \mathrm{d}x := f^*(h \, \mathrm{d}x)$$

for h a (hyper-)function. Explicit formula

$$(\mathcal{L}_f h)(x) = \sum_{f(y)=x} \frac{h(y)}{|Df(y)|}.$$

Thermodynamic limit reduction

If $\mathrm{d}\mu = h\,\mathrm{d}x$ we have (non-linear) dynamics

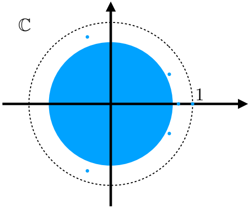
$$h_{n+1} = F(h_n) = \mathcal{L}_{f_{\int \phi h \,\mathrm{d}x}} h.$$

What can we say about F? The answer is in the theory of transfer operators. . .

Transfer operators

For many maps f with exponential decay of correlations:

- $\|\mathcal{L}_f\|_{L^1} = 1$.
- The set $\mathcal{M}$ of non-negative (hyper)functions integrating to one is invariant under $\mathcal{L}_f$.
- There is a smaller Banach space $\mathcal{B}$ on which $\mathcal{L}_f$ is *quasicompact*. (Probably many such $\mathcal{B}$) That is:
  - The spectral radius is 1, and
  - The essential (i.e. non-point) spectrum is confined to a disc of radius strictly less than 1.
- $f \mapsto \mathcal{L}_f$ has some differentiability properties, *only* if we consider $\mathcal{L}_f : \mathcal{B} \to \mathcal{B}^w \supset \mathcal{B}$ for appropriate weak space $\mathcal{B}^w$.



Transfer operators

If f is *very* nice (e.g. C^ω uniformly expanding):

- There is some Banach space $\mathcal{B}$ on which $\mathcal{L}_f$ is *compact* with spectral radius 1.
- In particular if the eigenvalues of $\mathcal{L}_f$ are given by $1 = |\lambda_1| \geq |\lambda_2| \geq \dots 0$, then (e.g. Bandtlow and Jenkinson '07)
$$|\lambda_k| \leq Ce^{-c\sqrt{k}}.$$
- $f \mapsto \mathcal{L}_f$ is C^∞ considering $\mathcal{L}_f : \mathcal{B} \to \mathcal{B}$.

Transfer operators

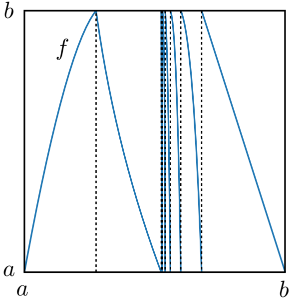
$$F(h) = \mathcal{L}_{f_{\int \phi h \, \mathrm{d}x}} h$$

From the last slide, we know:

- $F : \mathcal{B} \cap \mathcal{M} \circlearrowleft$ is well-defined and has nice compact images
- F is C^∞.
- $DF : \mathcal{B} \cap \{\phi : \int \phi = 0\} \circlearrowleft$ is compact.

Examples of nice f's

Uniformly expanding maps of the interval:



If f is (piecewise) C^{r+1} ($r > 0$), then $\mathcal{B} = C^r$ (among others).
If f is (piecewise) C^ω then $\mathcal{B} =$ some L^∞ Hardy space (i.e. bounded analytic functions on some complex set).

Numerics for nice f's

We can approximate transfer operators of unif. exp. maps extremely accurately using Chebyshev Galerkin methods (Wormell '19, Bandtlow and Slipantschuk '20).
In particular, we have the following estimates of $\mathcal{L}_f$ (hence F, DF, etc.) *in Hardy space $\mathcal{B}$ norm*:

$$\|\mathcal{L}_f - \underbrace{\mathcal{P}_N \mathcal{L}_f \mathcal{P}_N}_{\text{computable}}\|_{\mathcal{B}} \leq Ce^{-cN}.$$

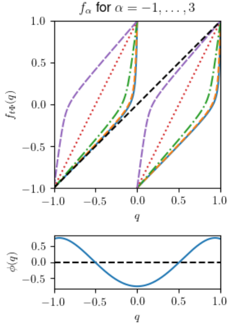
This plus compactness of $\mathcal{L}_f$ makes quite complex numerics possible!

Numerical example

Consider a family of coupled systems, parametrised by $t > 0$ regulating coupling strength:

$$\Phi_n = \frac{1}{M} \sum_{m=0}^{M} \phi(q_n^{(m)})$$

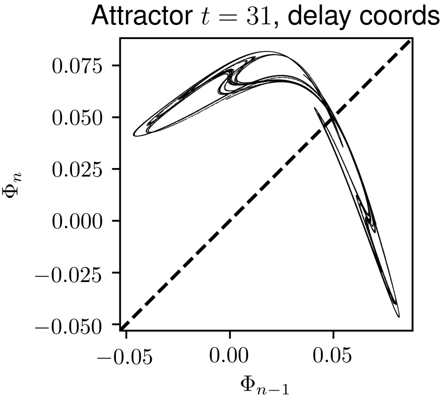
$$q_{n+1}^{(j)} = f_{t\Phi_n}(q_n^{(j)})$$



Form of f, ϕ chosen to induce unimodal dynamics in Φ_n (see W. and Gottwald '19).

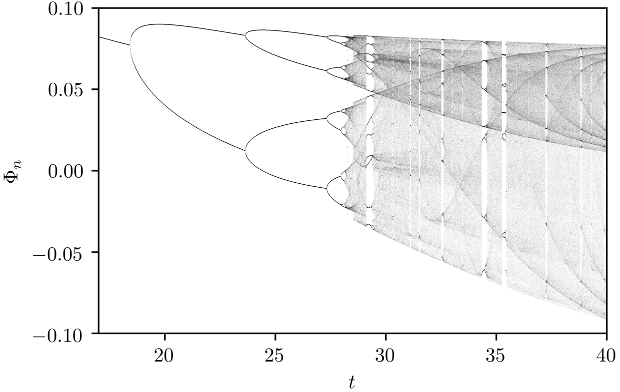
Numerical example

Hénon-like attractor at high coupling strengths:



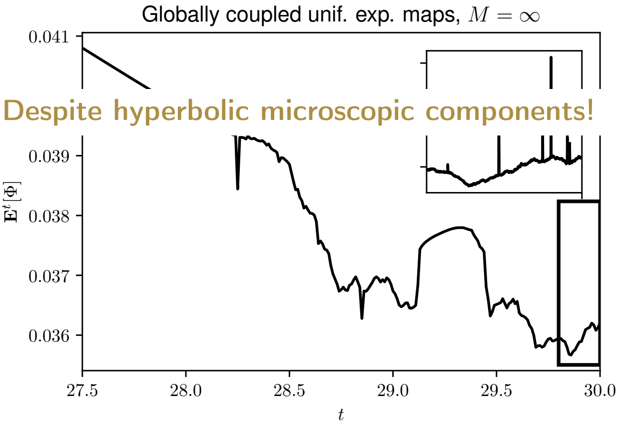
Numerical example

Hénon-like bifurcation structure:



Example

A failure of linear response:

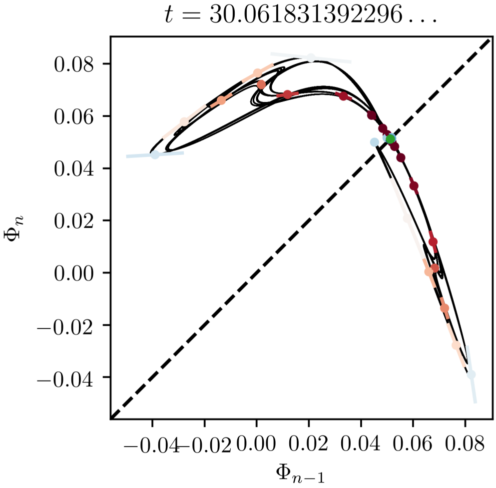


Despite hyperbolic microscopic components!

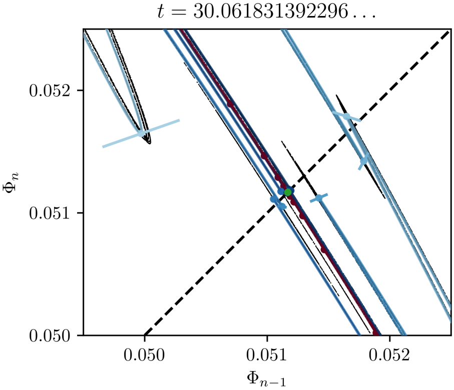
Is there really non-hyperbolicity afoot?

Homoclinic tangencies

We can use our fancy numerics to find a quadratic, tranverse homoclinic tangency. (Non-rigorous for now but provable.)



Homoclinic tangencies



$\Longrightarrow$ non-hyperbolicity in a mean field system! A blow for the chaotic hypothesis.

Homoclinic tangencies

Common caveat to CH: hyperbolicity occurs “generically” rather than universally.

But at least morally, we expect homoclinic tangencies on an open set of parameters! (Although these may not live on the attractors. . .)

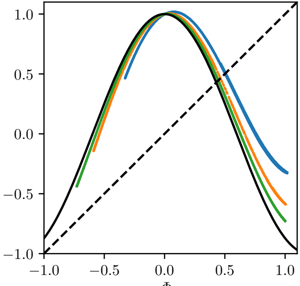
Arbitrary dynamics

Given any C^r function $g : [-1,1]^d \circlearrowleft$ and $\epsilon > 0$, there exists a mean-field system (with f Anosov diffeos and d-dimensional coupling function ϕ) such that

$$\Phi_{n+1} = g(\Phi_n) + \epsilon.$$

In fact, there is a map $F^\infty : \mathcal{B} \circlearrowleft$ semiconjugate to g such that for any $s < r$,

$$\|F - F^\infty\|_{C^s} \leq \epsilon$$



Arbitrary dynamics

In progress: "any C^k-open property of a diffeomorphism (e.g. existence of a blender) holds in a non-empty, C^∞-open set of globally coupled systems' thermodynamic limits".
Conclusion: cannot assume macroscale dynamics have hyperbolicity (or anything nice) *a priori*, at least in globally coupled systems.

Finite size

In practice, the number of coupled maps is likely to be finite, perhaps quite small.

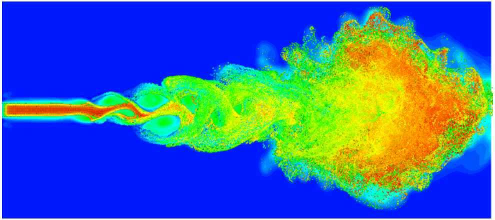


Figure: `http://mri-q.com`

What happens at finite size?

Finite size

Our mean-field has

$$\Phi_n = \frac{1}{M} \sum_{m=1}^{M} \phi(x_M^{(m)})$$

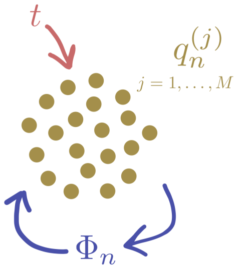
where the $x_n^{(m)}$ sample the thermodynamic measure limit μ_n.
By the central limit theorem we expect

$$\Phi_n = \int \phi \, \mathrm{d}\mu_n + \frac{1}{\sqrt{M}} \zeta_n,$$

where ζ_n is a Gaussian process. Combining this with

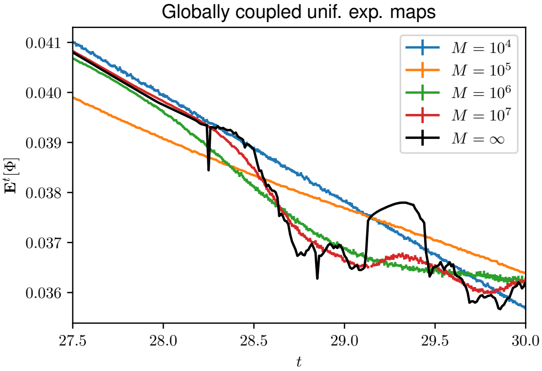
$$\mu_{n+1} = f^*_{\Phi_n} \mu_n$$

we obtain a **stochastic** process in our measure dynamics.



Finite size

Gaussian noise induces all the nice statistical properties that Anosov systems have, e.g. linear response:



So, in practice, what we see at the macroscale are (potentially non-hyperbolic) dynamics plus *noise*. Mystery solved??

Conclusion

Some questions for mean-field systems:

- How to treat lower-regularity systems (e.g. C^k subsystems, piecewise expanding?)
- What can we say about more realistic couplings (e.g. attractive/repulsive)?