On spurious detection of linear response in chaotic systems with finite time series

Presented by
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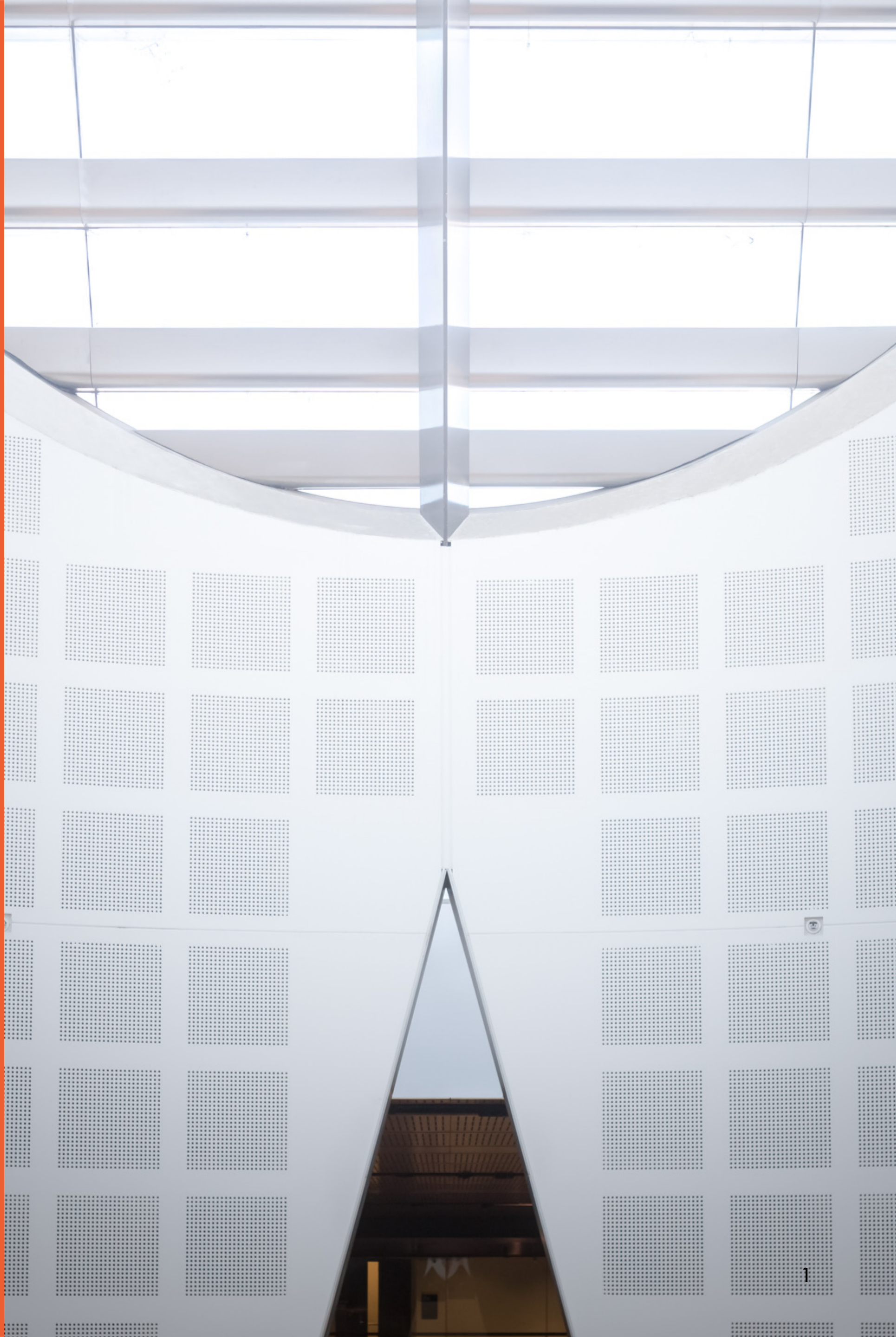

Statistical properties of chaotic systems

Consider a chaotic dynamical system

$$\dot{x} = f(x), \; x(0) = x_0.$$

Such systems typically have a physical measure μ, i.e., for any observable A and almost any initial value x_0,

$$\frac{1}{N}\int_0^N A(x(t))\,dt \xrightarrow{N\to\infty} \int_\Lambda A(x)\,d\mu =: \langle A \rangle$$

Time average

Phase space average

Statistical properties of chaotic systems

Consider a family of chaotic dynamical systems

$$\dot{x}_\epsilon = f(x_\epsilon, \epsilon), \; x_\epsilon(0) = x_0.$$

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$\boxed{\textbf{How does } \langle A \rangle_\epsilon \textbf{ vary with } \epsilon\textbf{?}}$

Linear response theory

Hypothesis: Maybe (for small epsilon) differentiably:

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Magic: *Both* coefficients can be calculated using only information about the statistics of the unperturbed system ($\epsilon = 0$) via formulae such as the *fluctuation-dissipation theorem*:

$$\langle A \rangle_0' = -\int_0^\infty \left\langle \frac{\mathrm{div}(\frac{\partial f}{\partial \epsilon}\mu)}{\mu} \; A \circ \Phi_t \right\rangle_0 \, dt$$

This has met with *qualified* success in climate science (work of A. Majda, A. Gritsun, V. Lucarini, Cooper & Haynes 13…)

Linear response theory - logistic map

But: chaotic maps may not have linear response.

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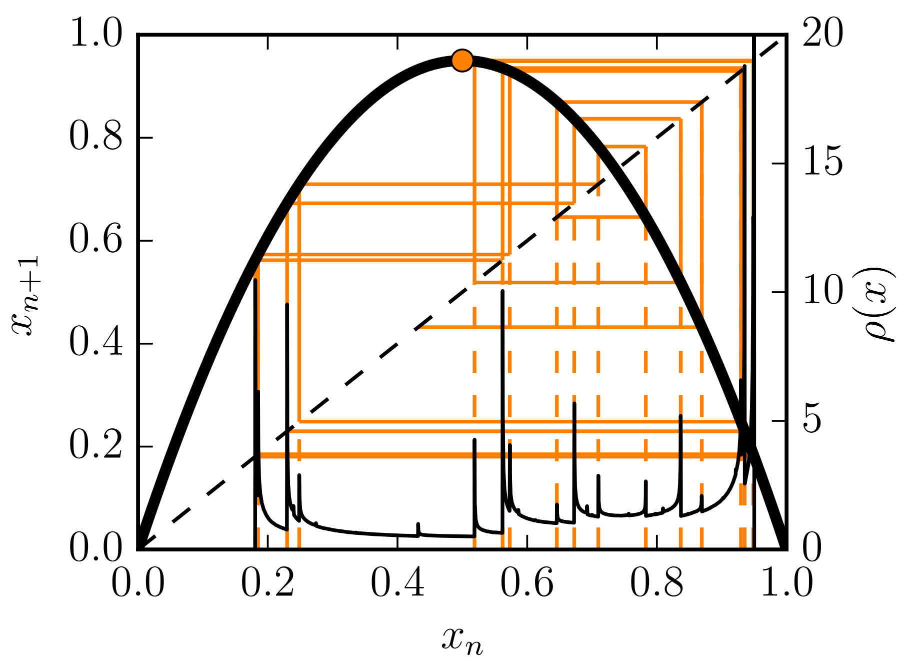
Case in point: the logistic map

$$x_{n+1} = (3.8 + \epsilon) x_n (1 - x_n)$$



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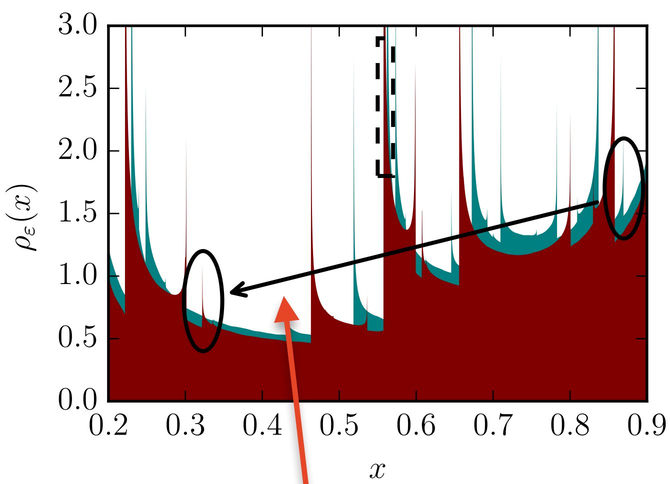
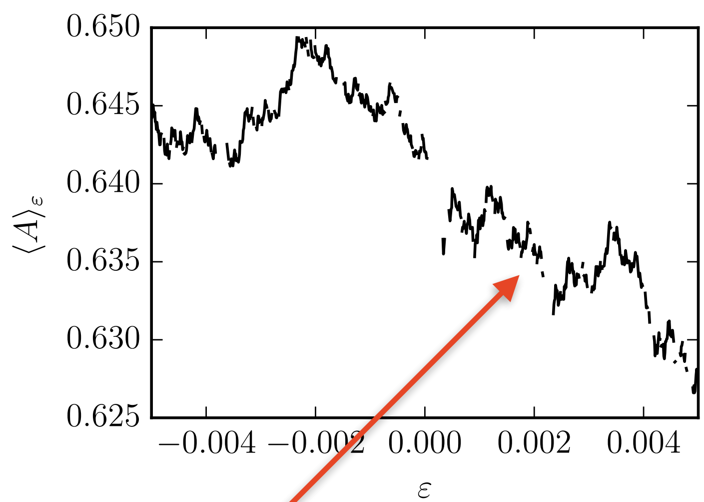


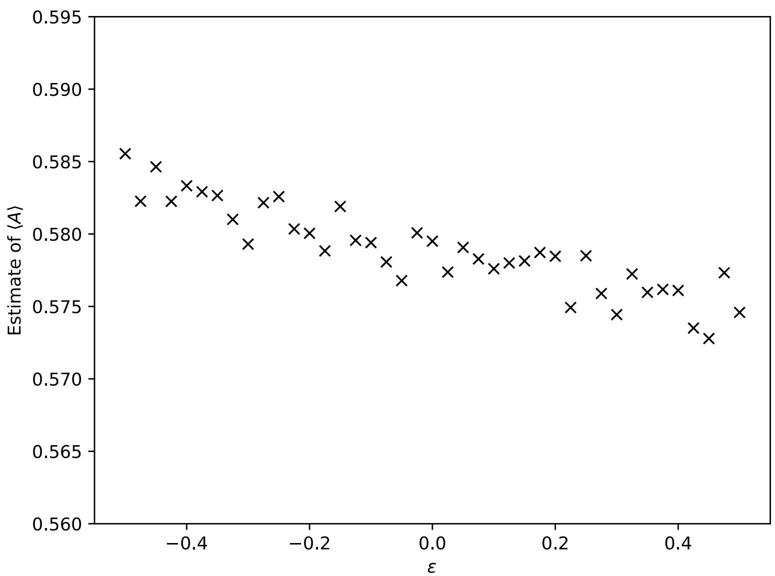


Small peaks move very fast

Holder-½ response
(plus periodic windows)

Linear response - time series

Physical measures are typically estimated by running long time series (i.e. Monte Carlo).



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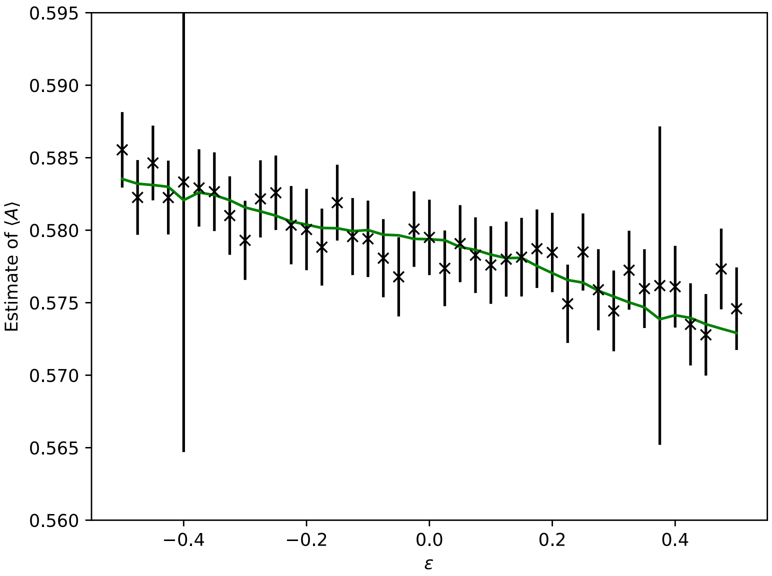

Perhaps it's a case of not enough data?

Test for linear response

How to see (statistically) if you have linear response:

Chaotic systems often obey a central limit theorem:

$$\bar{A}_{N,\epsilon} := \frac{1}{N}\int_0^N A(x_\epsilon(t))dt = \langle A\rangle_\epsilon + \frac{\sigma_\epsilon(A)}{\sqrt{N}}\xi,\ \ \xi \sim N(0,1)$$

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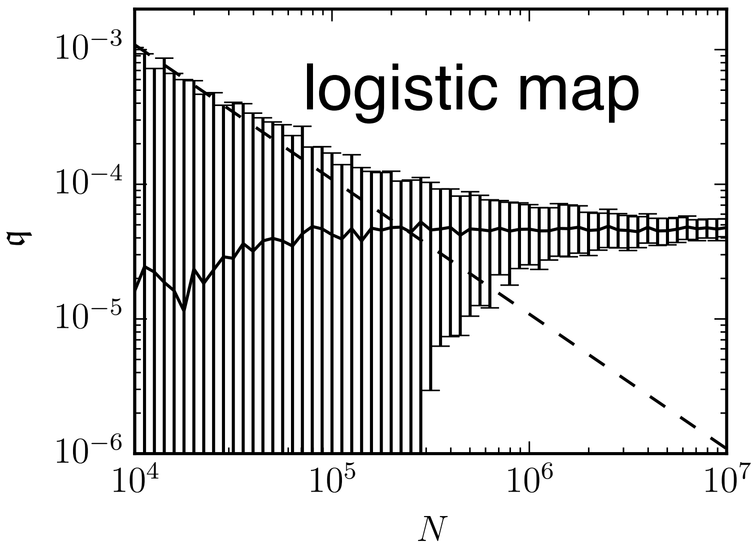
We run the system for different values of the parameter epsilon, and try and test the model for fit:

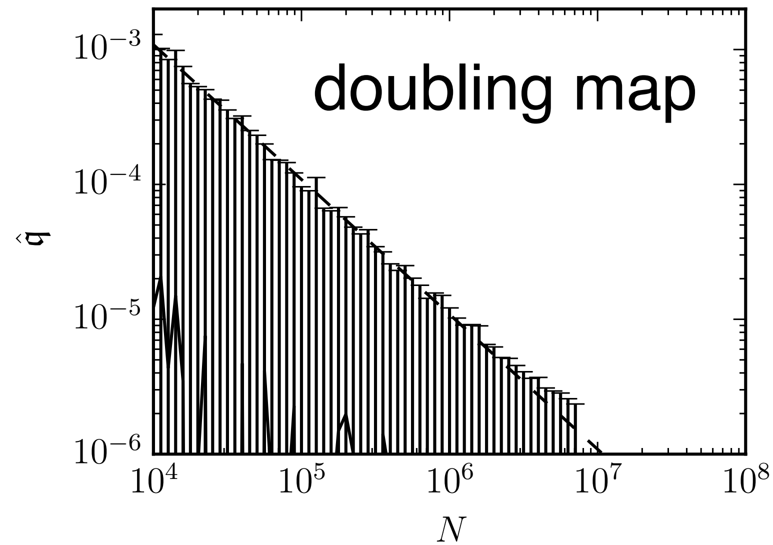
$$\bar{A}_{N,\epsilon_i} = a_0 + a_1\epsilon_i + \frac{\sigma_{\epsilon_i}(A)}{\sqrt{N}}\xi_i,\ \ \xi_i \sim N(0,1),\ \ i = 1,\dots,M$$

$$\chi^2 \sim \chi^2_{M-2},\ \ \mathfrak{q} = \frac{1}{N}(\chi^2 - \mathbb{E}\chi^2_{M-2})$$

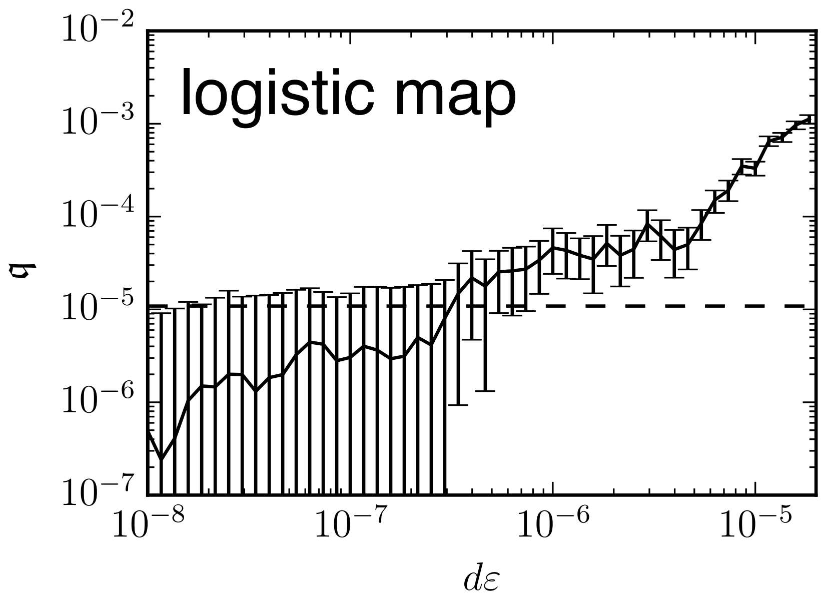
Effect of data size

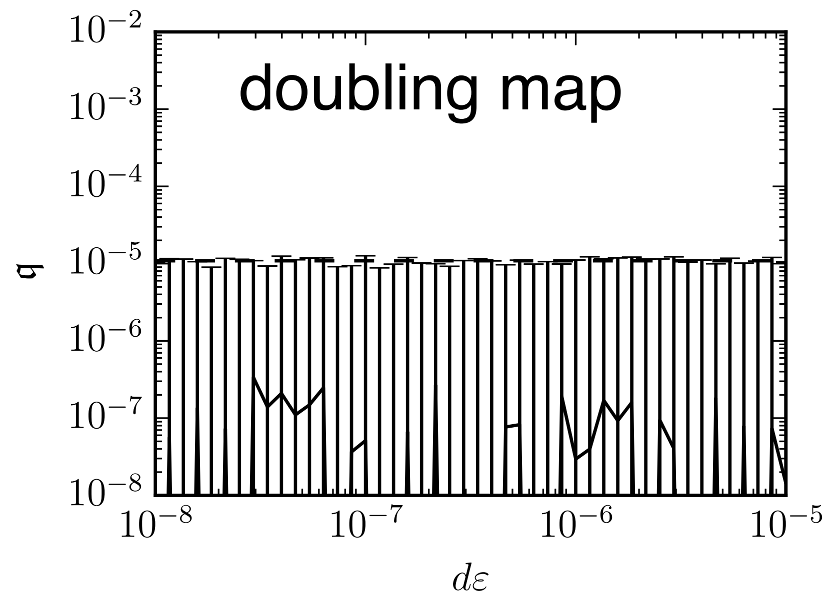
$$M = 20,\ \epsilon_{max} = 4 \times 10^{-5}$$





Effect of perturbation size

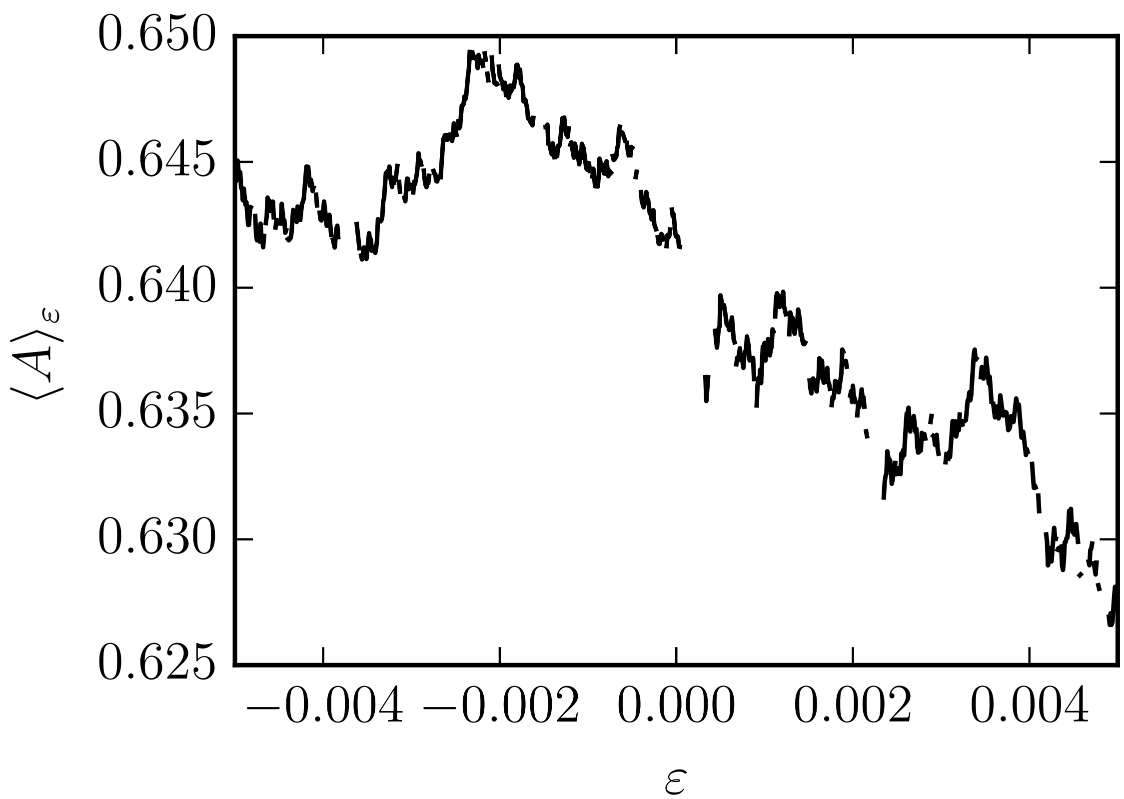




Conclusions from the logistic map

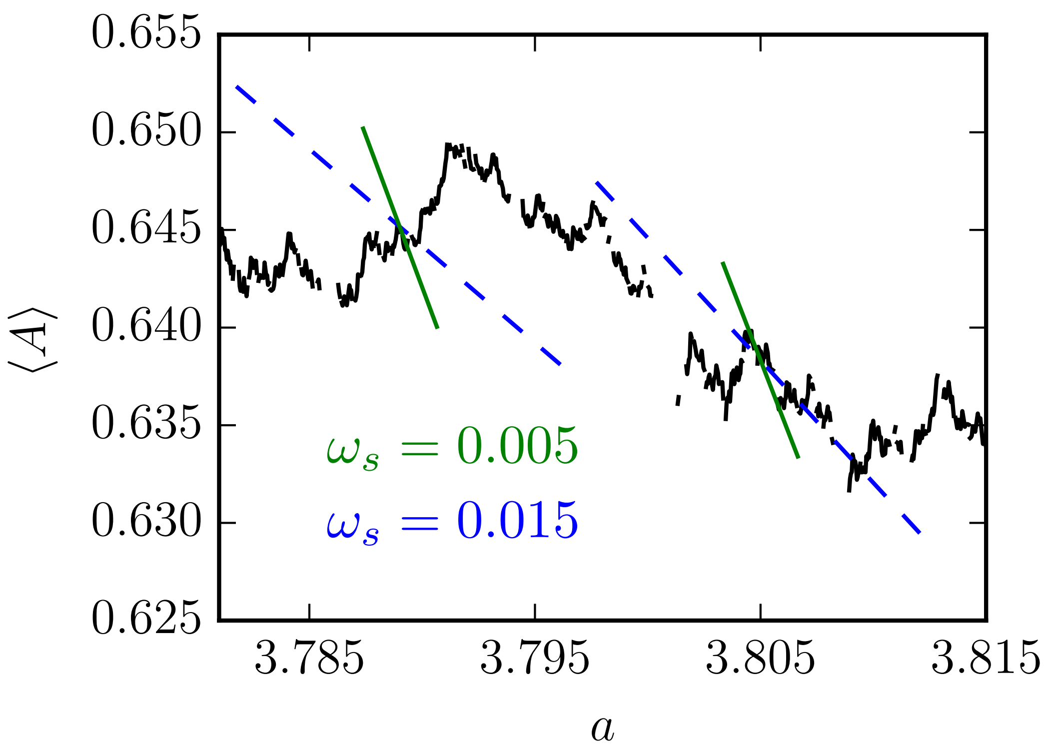
- It takes a lot of data ($NM \approx 50/\epsilon_{max}$) to reliably see the absence of linear response for global observables
- Using larger perturbations (i.e. bigger epsilon) makes this easier to see
- It is possible to reduce this using observables with localised support, but these require prior knowledge of the structure of the system

What about approximate linear fits?



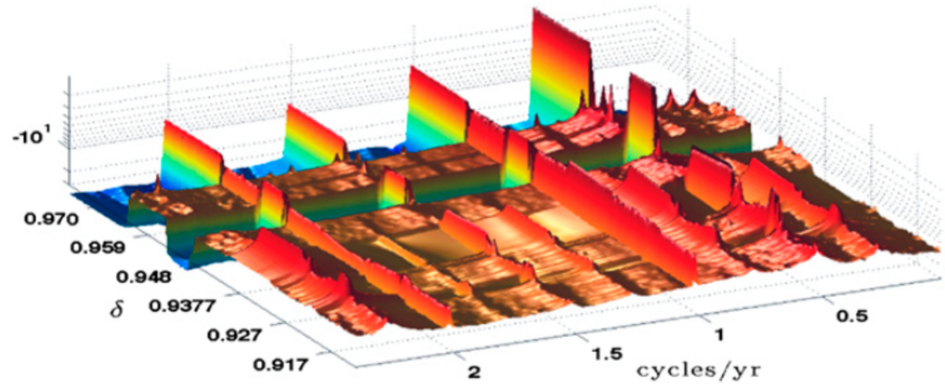
What about approximate linear fits?

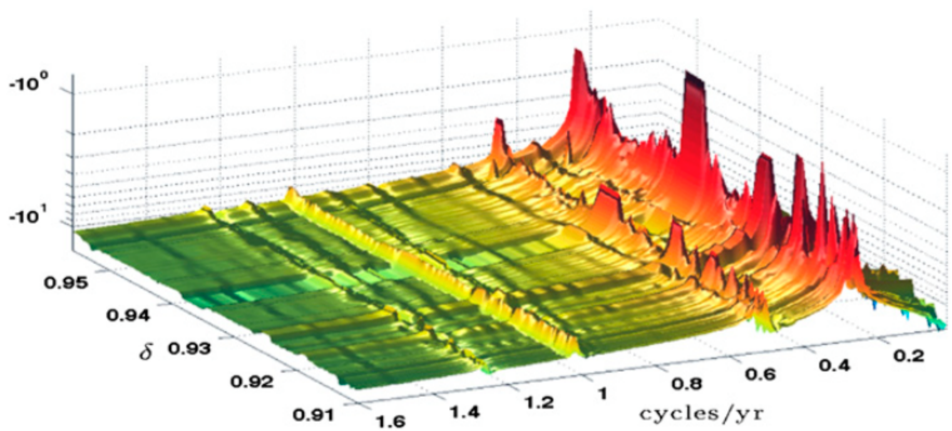
Trying to use fluctuation-dissipation theorem where linear response theoretically fails gives poorly-conditioned, meaningless results (for logistic map).



Further directions for research

- Application of linear response test to practically relevant systems
- Investigation of fluctuation-dissipation theorem
- Possible paths to linear response in complex systems (e.g. noise limits, strong versions of Gavalotti-Cohen hypothesis)





Chekroun et al. '14