Linear response in higher dimensions and mixing of Cantor sets

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Introduction

$f : \mathcal{M} \to \mathcal{M}$ chaotic, topologically mixing.

Suppose we have initial guess at state of system: measure μ on $\mathcal{M}$, and want to predict observable $A : \mathcal{M} \to \mathbb{R}$.
What happens in the long term:

$$\int_{\mathcal{M}} A \circ f^n \, \mathrm{d}\mu \xrightarrow{n \to \infty} ??$$

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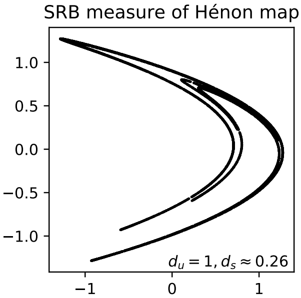
Most likely result: Sinai-Ruelle-Bowen (SRB) measure ρ_f and d_u positive Lyapunov exponents.



$\operatorname{supp} \rho_f$ is a union of d_u-dimensional unstable manifolds, on which conditional measure of ρ_f is abs. cts $\implies \dim \rho_f \geq d_u$.
Dimension of attractor cross-section $d_s = \dim \rho_f - d_u$.

Introduction

Why are these very important objects in chaotic dynamics?

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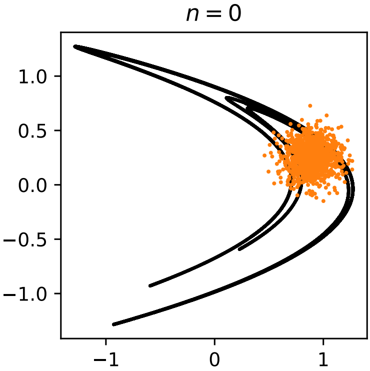
For many maps and regular (e.g. C^1) A, if μ is any absolutely continuous measure then future expectations of A converge exponentially fast to that of ρ_f

$$\frac{1}{|\mu|}\int_{\mathcal{M}} A \circ f^n \, \mathrm{d}\mu = \int A \, \mathrm{d}\rho_f + \mathcal{O}\left(e^{-cn}\right)$$

Thus, pushforward $\frac{1}{|\mu|} f_*^n \mu - \rho = \mathcal{O}(e^{-cn})$ in some Banach space dual.

That is, Lebesgue-absolutely continuous measures "mix" back to ρ_f:



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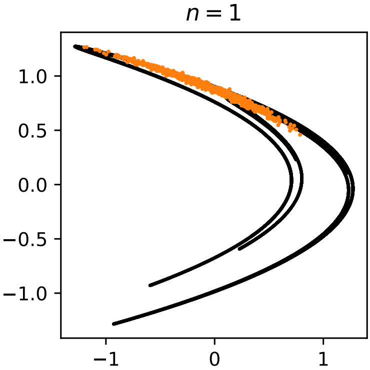

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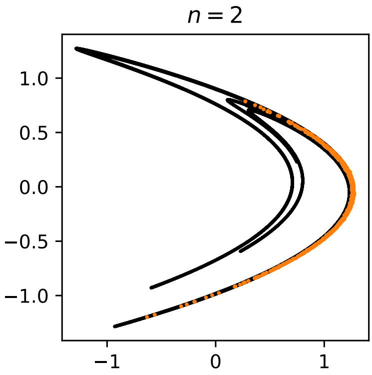

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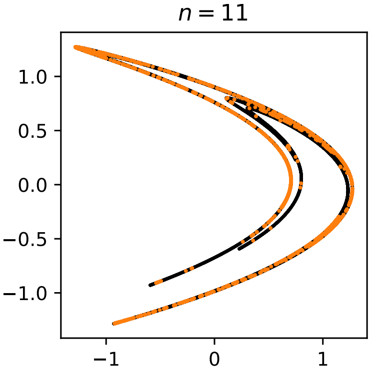

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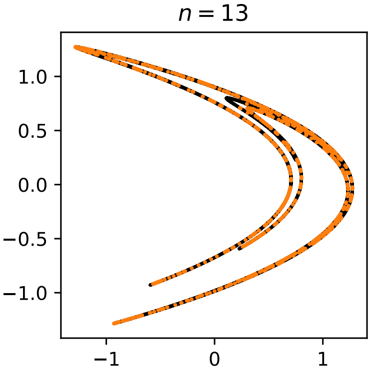

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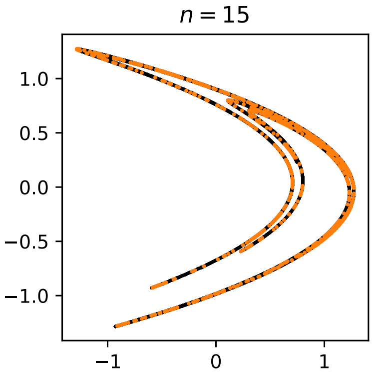

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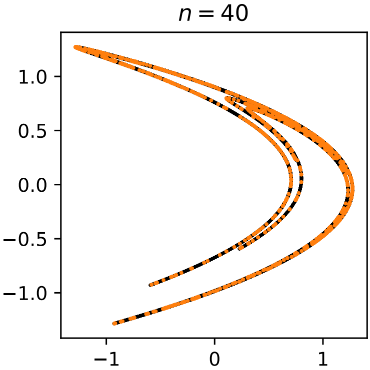


Introduction

This talk:

1. What larger classes of initial measures μ mix to ρ_f (exponentially)?
2. What might this tell us about the regularity of $f \mapsto \rho_f$? (= linear response problem)

Exponential mixing

Intuition: suppose μ is initial guess of system state, and $A \in \mathcal{B}$ is observable, then forecasts converge:

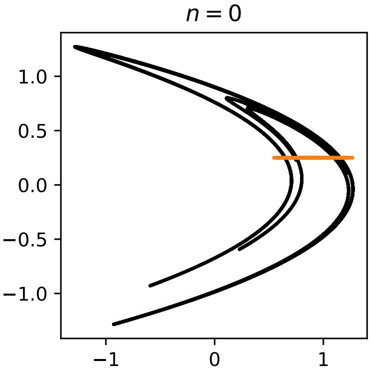
$$\frac{1}{|\mu|} \int_{\mathcal{M}} A \circ f^n \, \mathrm{d}\mu = \int A \, \mathrm{d}\rho_f + \mathcal{O}\left(e^{-cn} \|A\|_{\mathcal{B}}\right)$$

Equivalently, push-forward measure $f_*^n \mu \to |\mu| \rho_f$ in $\mathcal{B}^*$.
Transfer operator results (e.g. Gouëzel and Liverani '07, Baladi and Tsuji '07):

> *"If μ is conditionally absolutely continuous along $\geq d_u$-dimensional manifolds transverse to stable manifolds, with some regularity, then it exponentially mixes to ρ_f."*

Exponential mixing

Example 1: you could know some coordinates beforehand:



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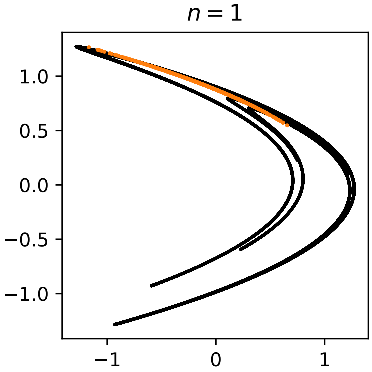

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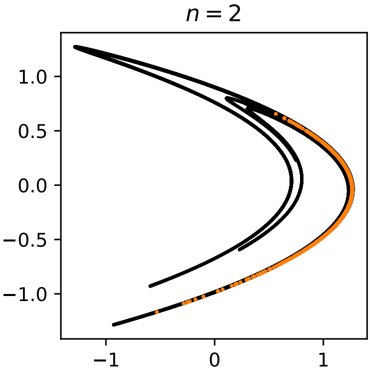

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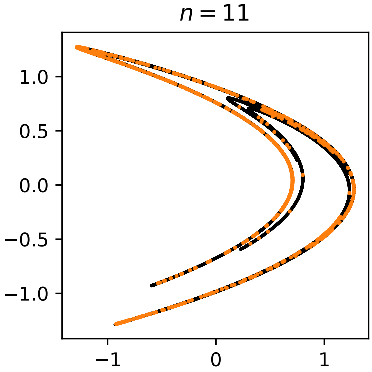

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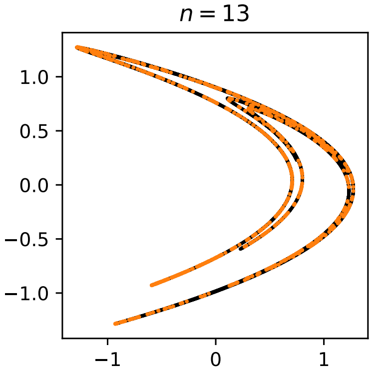

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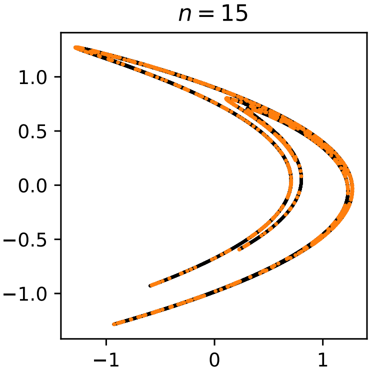

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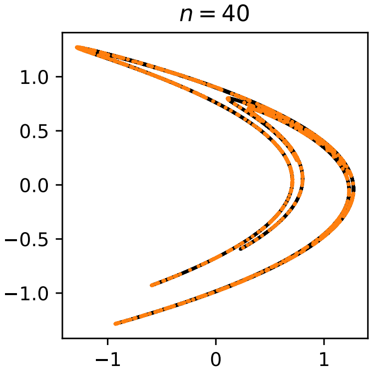
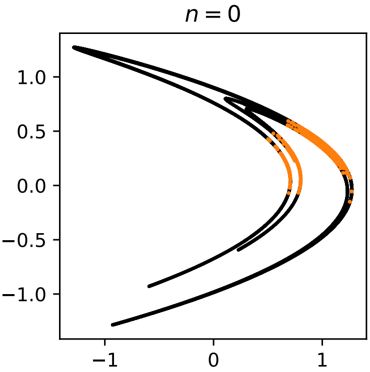


Exponential mixing

Or, you could have a re-weighted SRB measure: $\mu = B\rho_f$. Get classic decorrelation result:

$$\int A \circ f^n \, B \, \mathrm{d}\rho_f - \int A \, \mathrm{d}\rho_f \int B \, \mathrm{d}\rho_f = \mathcal{O}(e^{-cn} \|A\| \|B\|)$$



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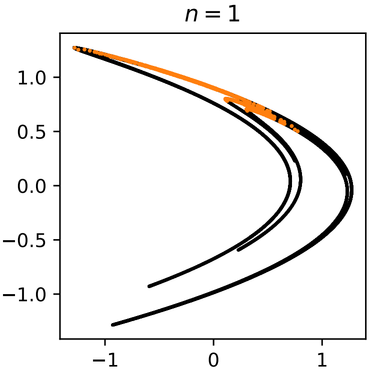

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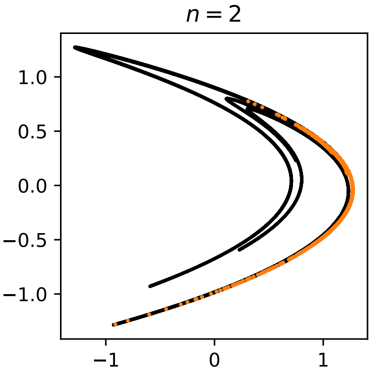

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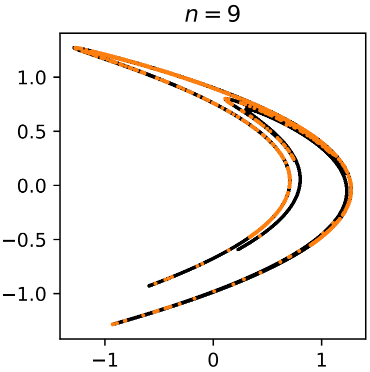

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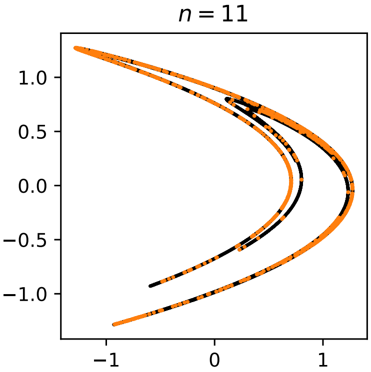

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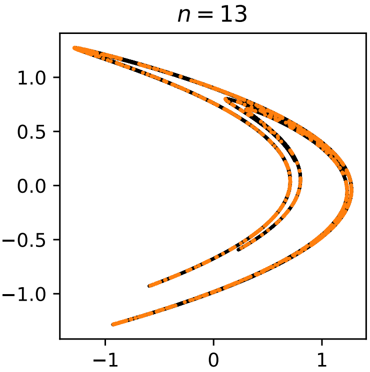

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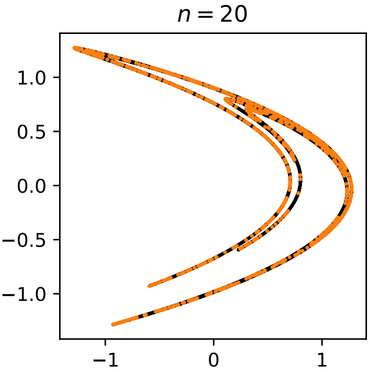

Exponential mixing

Transfer operator theory also usually gives second order mixing:

$$\int A \circ f^{m+n} \, B \circ f^m \, \mathrm{d}\mu_f - \int A \, \mathrm{d}\rho_f \int B \circ f^m \, \mathrm{d}\mu_f = \mathcal{O}(e^{-cn} \|A\| \|B\|)$$

uniformly in m.

Less regular measures

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Sometimes this doesn't work:

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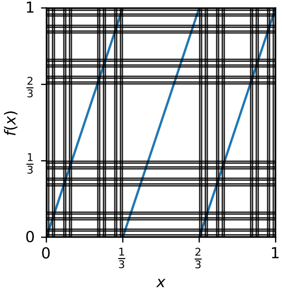
Example of #2:

- $f = 3x \mod 1$ tripling map, $\rho_f = \text{Leb}$
- $\operatorname{supp} \mu \subseteq C$ classical $\frac{1}{3}$-Cantor set.
- C is f-invariant and $\dim C < 1$.



Support of pushforwards of μ always $\subseteq C$...

Results

But what if you considered nice (but possibly still fractal) measures that don't obviously fail? Say, an invariant measure of a different map g.

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Recent work on "Fourier dimension" (Bourgain and others):

Theorem (Sahlsten and Stevens, '20)

If $f = kx \mod 1$, and μ is a Gibbs measure of a "totally non-linear" C^1 Markov expanding map g with Lipschitz potential, exponentially μ mixes to ρ_f.

Numerically appears to hold for other f...

Slice measures

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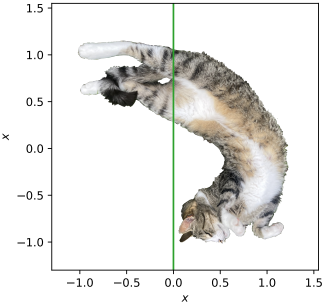
Let's try on the Lozi map

$$f(x, y) = (1 - a|x| + by, x), \;\; 0 < b < a - 1$$

Piecewise hyperbolic:



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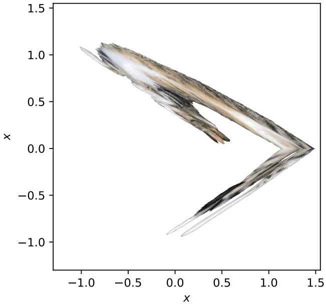
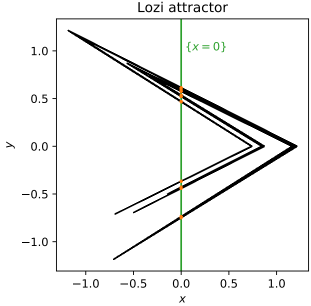


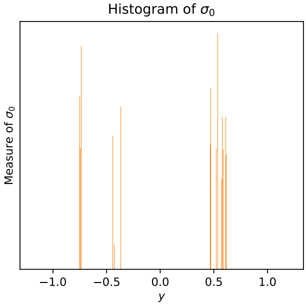
Slice measures

Dimensions: $d_u = 1$, $d_s \in (0,1)$.
Slice measure:

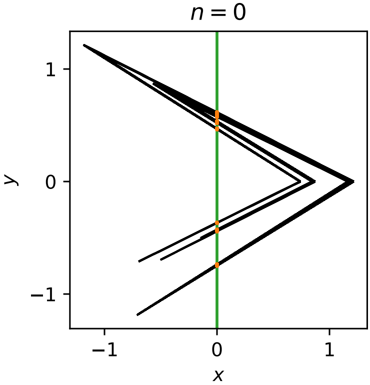
$$\sigma_c := \text{conditional measure of } \rho_f \text{ on } \{x = c\}.$$

Generically $\dim \sigma_c = d_s < 1$.





Slice slowly fills the full attractor:



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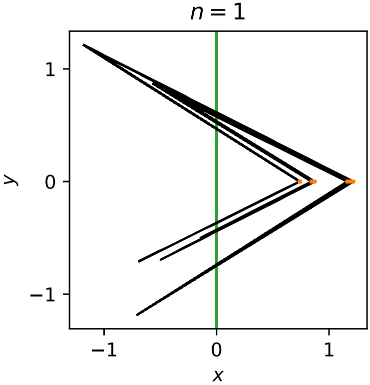

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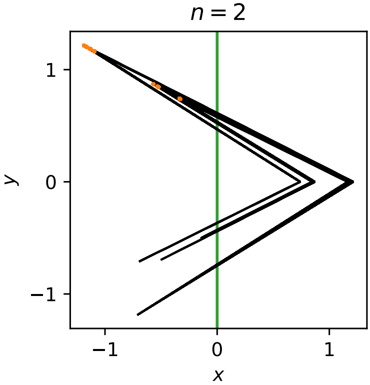

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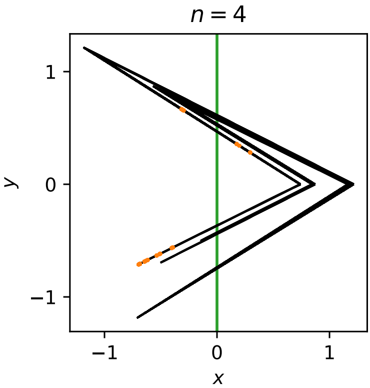

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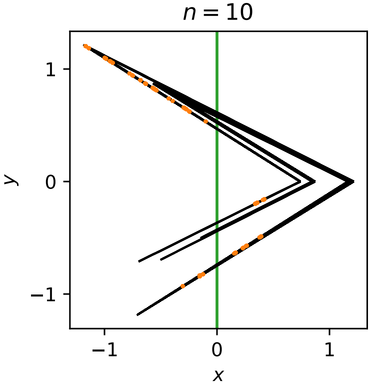

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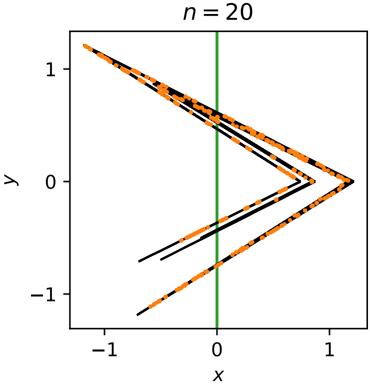

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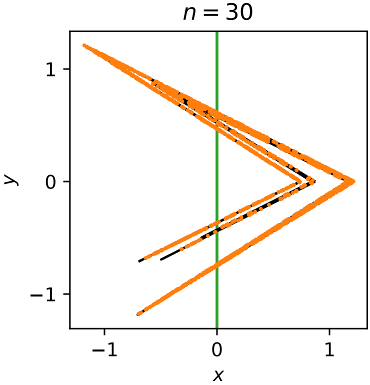
Slice slowly fills the full attractor:



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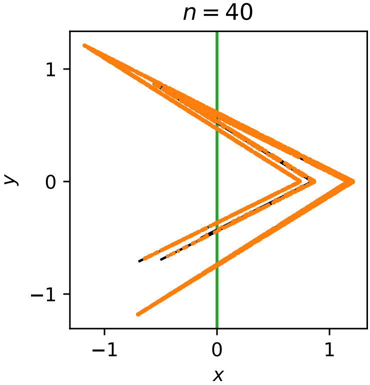
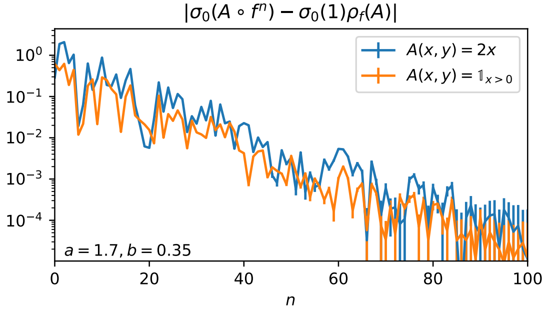


Push-forwards of σ_0 numerically difficult to observe accurately!
However:

Theorem

The following chart was made with an accurate Monte Carlo sample of σ_0:



Q: How do you get a good sample from $f_*^n \sigma_c$?

Slice measures

Q: How do you get a good sample from $f_*^n \sigma_c$? A: get a very good sample from σ_c.

- Lozi map is affine, so at any point p the local unstable manifold $W^u_{\mathrm{loc}}(p)$ is an interval $\implies$ easy to compute
- We can sample from σ_c via $\{W^u_{\mathrm{loc}}(f^n(p))\} \cap \{x = c\}$ for some $p \sim \rho_f$.
- Of course, doing this in a way that excludes machine error requires more tricks. . .

Response theory

Q: What is the regularity of $f \mapsto \rho_f$?

Response theory

More specific Q: if

$$f_\varepsilon(x) = f(x) + \varepsilon X(f(x)) + o(\varepsilon),$$

what is regularity of the “response” $\varepsilon \mapsto \int A \,\mathrm{d}\rho_{f_\varepsilon}$?

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If it is C^1, then $\exists$ a formula to compute the derivative at $\varepsilon = 0$ from f dynamics.

$\implies$ easy prediction of statistical response to dynamical perturbations

Response theory

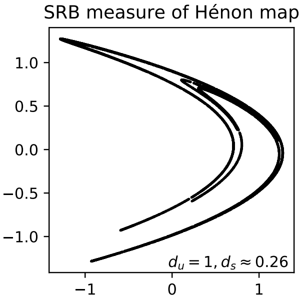
When is the response differentiable for generic perturbations?

- ✓ Stochastic systems (Hänggi '78, Hairer and Majda '10)
- ✓ Smooth(!) hyperbolic systems (Ruelle '97)
- × Logistic maps: $C^{1/2-\delta}$ at best (Baladi and Smania '12, '14)
- × 1D piecewise expanding maps, *e.g.* $f(x) = 1 - a|x|$: $C^{1-\delta}$ at best (Mazzalena '07, Baladi '07)
- ? Piecewise hyperbolic diffeomorphisms, *e.g.* Lozi map
- ? Non-hyperbolic dissipative diffeomorphisms *e.g.* systems of real interest

Conjecture (Ruelle '19)

Generic non-hyperbolic smooth diffeomorphisms have at least $C^{d_s+1/2-\delta}$ response (formally).

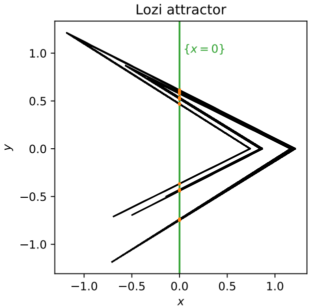
NB: for logistic map, $d_s = \dim \rho_f - d_u = 0$ so $C^{1/2-\delta}$ response.



Very hard to impossible rigorous setting. Need simpler example. . .

Theorem

Lozi maps with second-order mixing of σ_0 to ρ_f formally have a linear (C^1) response.



Lozi map

Why?

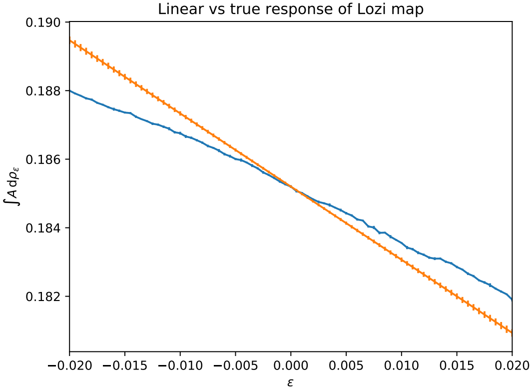
- Linear response is a correlation with derivative of the measure:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_{f_\varepsilon}\bigg|_{\varepsilon=0} = -\sum_{n=0}^{\infty}\int A\circ f^n\,\nabla\cdot(X\mathrm{d}\rho)$$

- Possible failure points are where measure is not regular enough in unstable direction (e.g. jumps in density)
- Measure irregularities are localised to the orbit of singular points in map
- Singular set of the Lozi map is $\{x = 0\}$.

Lozi map

Following Ruelle's work, expect $C^{d_s+1-\delta}$ response, i.e. low-order:



Conclusion

- Mixing of non-smooth measures to SRB measure an interesting phenomenon/problem
- Results obtained should generalise to other piecewise hyperbolic systems
- Will be key to understanding existence of linear response in higher dimensions