Conditional decay of correlations and linear response

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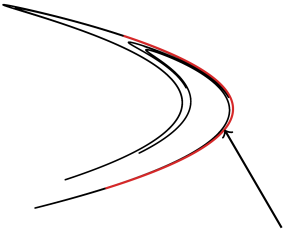
Setting

A smooth chaotic dynamical system $T : M \to M$ endowed with invariant *SRB measure* ρ.



conditional measure abs. cts. wrt Lebesgue

e.g.: if $T(x) = kx \mod 1$ then $\rho =$ Lebesgue measure.

SRB measure is the physically important invariant measure!

Convergence of measures

A fundamental desideratum:

Exponential decay of correlations

There exists $\xi < 1$, C such that if $A, B \in C^1(M, \mathbb{R})$ then

$$\left| \int_M A \circ T^n \, B \, \mathrm{d}\rho - \int_M A \, \mathrm{d}\rho \int_M B \, \mathrm{d}\rho \right| \leq C \|A\|_{C^1} \|B\|_{C^1} \xi^n.$$

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i.e. $T_*^n \mu \to (\int \mathrm{d}\mu)\rho$ exponentially quickly in $(C^1)^*$

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A Question: *what other measures μ converge to the SRB measure?*

Convergence of measures

Obviously some measures don't do this:

- Delta function: $\mu = \delta_{x_0}$. Then $T_*^n \mu = \delta_{T^n(x_0)} \neq \rho$.
- Another invariant measure: $\mu \in \mathcal{M}(M, T)$, $\mu \neq \rho$. Then $T_n^* \mu = \mu \neq \rho$.
- "Adversarial" constructions: $T(x) = 2x \mod 1$, $X_n = \text{Bernoulli}(p_n)$ ind., $\mu \sim_\ell \sum_{n=1}^{\infty} 2^{-n} X_n$. Then $\int \mathbb{1}(T^n(x) > \frac{1}{2}) \, \mathrm{d}\mu(x) = p_n$.

Convergence of measures

But clearly some do (apart from $B\rho$ with $B \in C^\alpha$):

Theorem

If $T(x) = kx \mod 1$ *and* $\dim_F \mu > 0$*, then for all* $\delta < \dim_F \mu$*,*

$$\left| \int_0^1 A \circ T^n \, \mathrm{d}\mu - \int_0^1 A \, \mathrm{d}\rho \int_0^1 \mathrm{d}\mu \right| \leq C k^{-\delta n} \|A\|_{C^1}.$$

Sketch of proof:

$$\int_0^1 A \circ T^n \, \mathrm{d}\mu = \sum_{l=-\infty}^{\infty} \hat{A}_{k^n l} \hat{\mu}_{-l} = \hat{A}_0 \hat{\mu}_0 + \mathcal{O}((k^n \cdot 1)^{-\delta})$$

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But clearly some do (apart from $B\rho$ with $B \in C^\alpha$):

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If $T(x) = kx \mod 1$ and $\dim_F \mu > 0$, then for all $\delta < \dim_F \mu$,

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Convergence of measures

Conjecture

Generic positive-dimensional, "nicely" generated probability measures μ *should have* $T_*^n \mu \to \rho$ *exponentially in* $(C^1)^*$ *if* T *is smooth and exponentially mixing.*

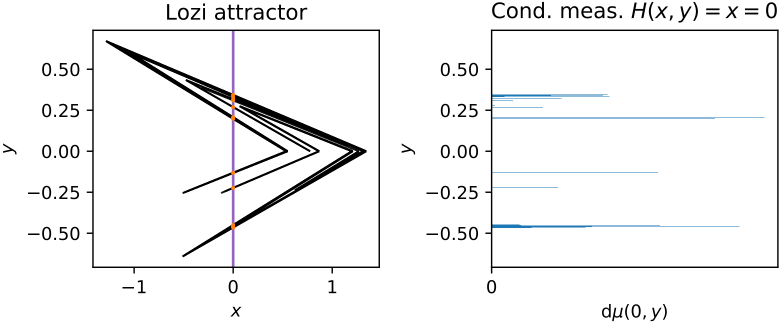
Specific examples?

Conditional measures

For some function $H : M \to \mathbb{R}$ take

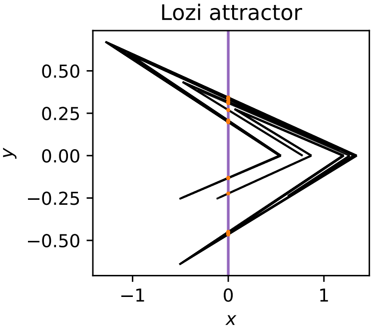
$$\mathrm{d}\mu(x) = \lim_{\eta \to 0} \frac{\mathbb{1}(|H(x)| < \eta)}{2\eta} \,\mathrm{d}\rho(x) = \text{“}\delta(H(x))\,\mathrm{d}\rho(x)\text{”}$$

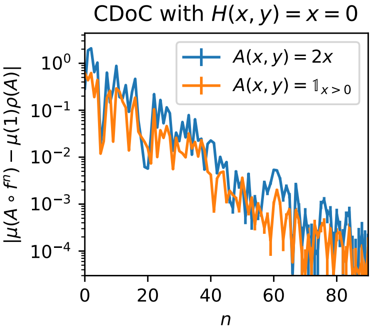
so $\mu \propto$ conditional measure $\rho(x \mid H(x) = 0)$.



Conditional decay of correlations

Conditional decay of correlations (CDoC): $T_*^n B\mu \to \rho \int B \mathrm{d}\mu$ for all "nice" B.
Looks true for Lozi maps (pw affine approximations of Hénon):





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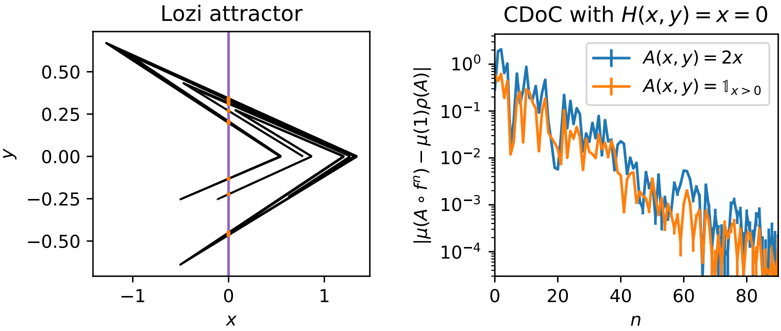

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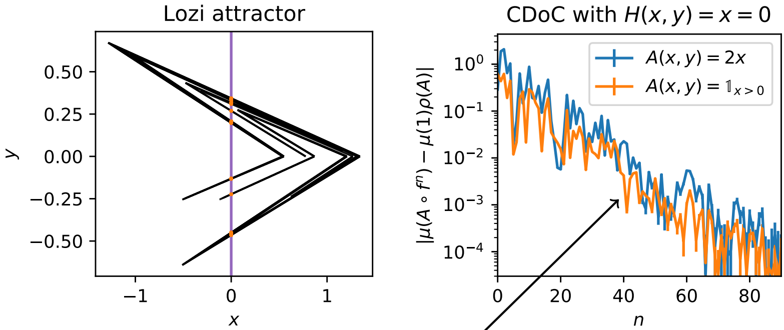


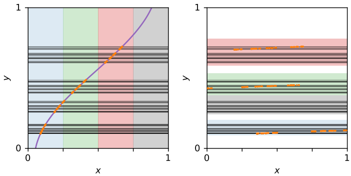
Rigorously validated!

Conditional decay of correlations

Study map on $[0,1]^2$:

$$T(x,y) = (kx \mod 1, v_{\lceil kx \rceil}(y))$$

with v_i non-overlapping contractions.



Theorem (W. '22)

The conditional measure $\mu = \rho(\cdot \mid \psi(y) - x = 0)$ is well-defined. Furthermore, if $\psi' \neq 0$ and either

- *The v_i are analytic and totally nonlinear*
- *Each $v_i(x) = \alpha x + \beta_i$ and $\psi'' \neq 0$*

then there exists $\xi < 1$ such that

$$\left| \int A \circ T^n \, \mathrm{d}\mu - \int A \, \mathrm{d}\rho \int \mathrm{d}\mu \right| \leq C\xi^n \|A\|_{C^1}.$$

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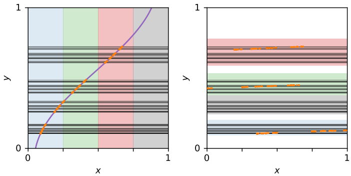

Theorem (W. '22)

The conditional measure $\mu = \rho(\cdot \mid \psi(y) - x = 0)$ is well-defined. Furthermore, if $\psi' \neq 0$ and either

- *The v_i are analytic and totally nonlinear (Sahlsten and Stevens '20)*
- *Each $v_i(x) = \alpha x + \beta_i$ and $\psi'' \neq 0$ (Mosquera and Schmerkin '18)*

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$$\left| \int A \circ T^n \, \mathrm{d}\mu - \int A \, \mathrm{d}\rho \int \mathrm{d}\mu \right| \leq C \xi^n \|A\|_{C^1}.$$

Big application: linear response theory

Smooth family of perturbations:

$$T_\varepsilon(x) = T(x) + \varepsilon X(T(x)) + o(\varepsilon)$$

Each T_ε has SRB measure ρ_ε.

The "linear response" $\frac{\mathrm{d}\rho_\varepsilon}{\mathrm{d}\varepsilon}$ can be computed from T, X, ρ, *if it exists. . .*

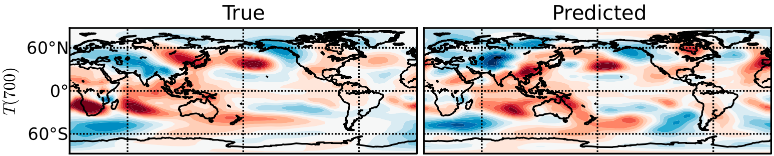
Results in linear response

Theoretically:

- Linear response for Axiom A
- No linear response for logistic maps
- No (rigorous) clue about 99.9% of real systems

Practically:

- 50+ years of success in calculating linear responses in physics and climate science



Fuchs, Hernandez and Sherwood 2014

Linear response formula

Why?

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_\varepsilon\bigg|_{\varepsilon=0} = -\sum_{n=0}^{\infty}\int A\circ T^n\ \frac{\mathrm{d}\nabla\quad\cdot(X\mathrm{d}\rho)}{\mathrm{d}\rho}\ \ \mathrm{d}\rho$$

Linear response formula

Why?

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_\varepsilon\Big|_{\varepsilon=0} \sim -\sum_{n=0}^{\infty}\int A\circ T^n \underbrace{\frac{\mathrm{d}\nabla^{u}\cdot(X\mathrm{d}\rho)}{\mathrm{d}\rho}}_{\text{Typically a density}}\,\mathrm{d}\rho$$

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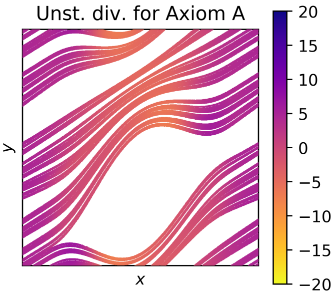
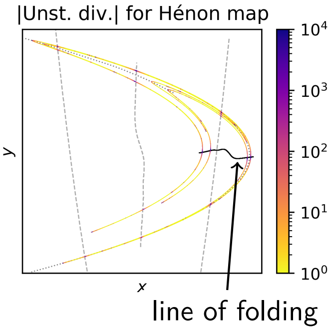
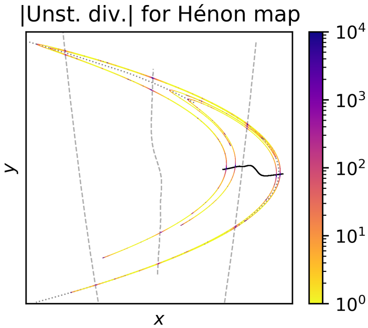




Lozi map

Lozi map simplifies Hénon map:



$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 - ax^2 + y\\by\end{pmatrix}$$

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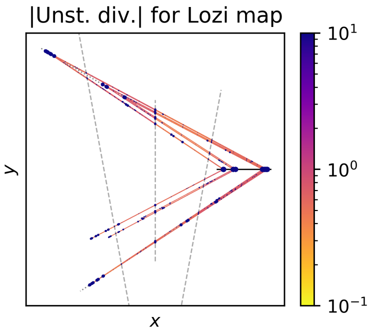

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - a|x| + y \\ by \end{pmatrix}$$

Linear response

We know that $\varepsilon \mapsto \rho_\varepsilon$ is C^α for all $\alpha < 1$ for the Lozi map.

Theorem (W. '22)

Conditional decay of correlations (slightly extended) for the Lozi map implies that the linear response formula converges.

What could understanding conditional decay of correlations give us?

- Proofs of regularity of $\varepsilon \mapsto \rho_\varepsilon$
  - Ruelle ('18) conjectures that large stable dimension of SRB measure improves regularity
- Ideas for better computation of linear response in real systems

Summary

Conditional decay of correlations

$$\text{If } \mathrm{d}\mu(x) = \mathrm{d}\rho(x \mid H(x) = 0),$$

$$\text{does } \int A \circ T^n \,\mathrm{d}\mu \xrightarrow{n\to\infty} \int A \,\mathrm{d}\rho\,?$$

- CDoC: a fractal *and* dynamical question
  - Proof for toy model via Fourier dimension
  - Credible numerics for other systems

  `arXiv:2206.09292`
- CDoC $\implies$ linear response for Lozi map and likely more:
  `arXiv:2206.09291`

Let's talk: `wormell@lpsm.paris`