Prediction from perfect partial observations and linear response

Caroline Wormell

Sorbonne Université, CNRS

June 30, 2022

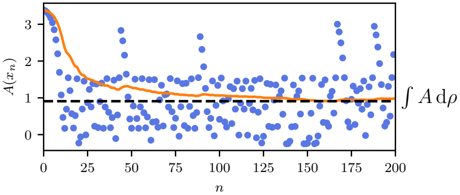
Setting

Consider a mixing chaotic dynamical system $x_n = T(x_{n-1})$, with a physical invariant measure ρ.

Setting

Consider a mixing chaotic dynamical system $x_n = T(x_{n-1})$, with a physical invariant measure ρ.
The physical measure encodes long-term ergodic behaviour of x_n.
Mathematically, for observables A and Lebesgue-a.e. x_0,

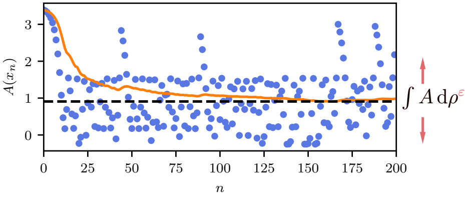
$$\frac{1}{N}\sum_{n=0}^{N-1} A(x_n) \xrightarrow{N\to\infty} \int A(x)\,\mathrm{d}\rho(x) = \text{“}\mathbb{E}[A]\text{”}$$



Setting

Consider a smooth family ofmixing chaotic dynamical systems $x_n = T^{\varepsilon}(x_{n-1})$, with physical invariant measures ρ^{ε}.
The physical measures encode long-term ergodic behaviour of x_n.
Mathematically, for observables A and Lebesgue-a.e. x_0,

$$\frac{1}{N}\sum_{n=0}^{N-1} A(x_n) \xrightarrow{N\to\infty} \int A(x)\,\mathrm{d}\rho^{\varepsilon}(x) = \text{“}\mathbb{E}^{\varepsilon}[A]\text{”}$$



Linear response theory

$$\text{“}\mathbb{E}^{\varepsilon}[A]\text{”} := \int A(x)\,\mathrm{d}\rho^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho^{\varepsilon}$*?*
(e.g. for Taylor approximations)

Linear response theory

$$\text{“}\mathbb{E}^{\varepsilon}[A]\text{”} := \int A(x)\,\mathrm{d}\rho^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho^{\varepsilon}$*?* (e.g. for Taylor approximations)

Linear response theory

$$\text{“}\mathbb{E}^{\varepsilon}[A]\text{”} := \int A(x)\,\mathrm{d}\rho^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho^{\varepsilon}$*?*
(e.g. for Taylor approximations)

. . . supposing differentiability of $\varepsilon \mapsto \rho^{\varepsilon}$!

Linear response theory

$$“\mathbb{E}^{\varepsilon}[A]” := \int A(x)\,\mathrm{d}\rho^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho^{\varepsilon}$*?* (e.g. for Taylor approximations)

…supposing differentiability of $\varepsilon \mapsto \rho^{\varepsilon}$!

When and why do we have differentiability?

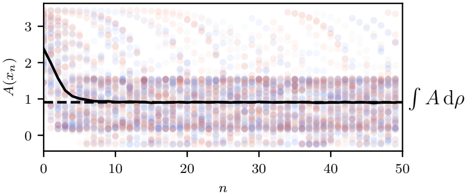
Decay of correlations

Classic result in dynamics (usually expect to be true):

Exponential decay of correlations

There exists $\theta < 1$, C such that if $A, B \in C^1(M, \mathbb{R})$ then

$$\left| \int_M A \circ T^n \, B \, \mathrm{d}\rho - \int_M A \, \mathrm{d}\rho \int_M B \, \mathrm{d}\rho \right| \leq C \|A\|_{C^1} \|B\|_{C^1} \theta^n.$$



Decay of correlations

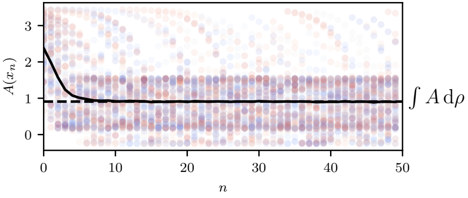
Classic result in dynamics (usually expect to be true):

Exponential decay of correlations

There exists $\theta < 1$, C such that if $\mu = B\rho$ and $A, B \in C^1(M, \mathbb{R})$ then

$$\left| \int_M A \circ T^n \, \mathrm{d}\mu - \int_M A \, \mathrm{d}\rho \int_M \mathrm{d}\mu \right| \leq C \|A\|_{C^1} \|B\|_{C^1} \theta^n.$$

i.e. $T_*^n \mu \to (\int \mathrm{d}\mu)\rho$ exponentially quickly in $(C^1)^*$



Decay of correlations

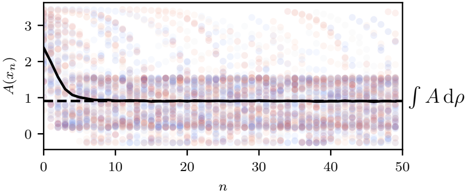
Classic result in dynamics (usually expect to be true):

Exponential decay of correlations

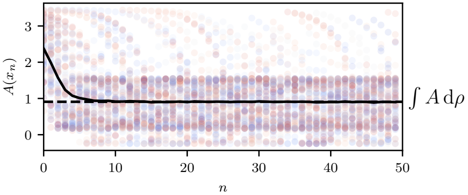
There exists $\theta < 1$, C such that if $\mu = B\rho$ and $A, B \in C^1(M, \mathbb{R})$ then

$$\left| \int_M A \circ T^n \, \mathrm{d}\mu - \int_M A \, \mathrm{d}\rho \int_M \mathrm{d}\mu \right| \leq C \|A\|_{C^1} \|B\|_{C^1} \theta^n.$$

i.e. $\underbrace{T_*^n}_{\text{transfer operator}} \mu \to (\int \mathrm{d}\mu)\rho$ exponentially quickly in $(C^1)^*$



Decay of correlations



- Diff'ble function B in $\mu = B\rho$ can represent some *imperfect* information of the system's state.
- Decay of correlations = this information will be lost over time
- How much can we improve our knowledge if we improve our information?

Sense of talk

Two questions:

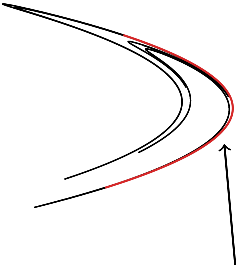
A. Prediction using perfect (or very good) observations.

B. Linear response (for general systems)

We will show that the two are governed by the same phenomenon: "*conditional* decay of correlations".

A note on physical measures

A physical measure is typically an SRB measure, i.e. has the following geometrical property:



conditional measure has a density

Prediction: setup

System $x_n \in M \subseteq \mathbb{R}^d$ undergoing dynamics $x_{n+1} = T(x_n)$.
At certain times t we make an *observation*

$$z_t = H(x_t) + \xi_t \in \mathbb{R}^e$$

$H : \mathbb{R}^d \to \mathbb{R}^e$
$e \ll d$ typically: **partial obs**

noise $\sim \mathcal{N}(0, \sigma^2 I)$

Prediction: filter

- Given $z_t = H(x_t) + \xi_t$ we want to estimate x_n for $n \geq t$.
- Standard to do this probabilistically, i.e. distribution of likely x_n
- Many options: particle filters, Kalman filters, grid methods...
- All approximate a *Bayesian filter*

Bayesian filter: assimilation step

Let's do *one* assimilation step at $t = 0$.

- Suppose we are given prior distribution of x_0 μ_0^-.
- Then we can assimilate observation z_0 to get posterior:

$$\mathrm{d}\mu_0^+(x) = \mathbb{P}\left(X_0 = x \mid H(X) + \xi = z_0\right),\ X_0 \sim \mu_0^-$$

- If ξ has pdf p_σ then Bayes' law says

$$\mathrm{d}\mu_0^+(x) = \ Z_\sigma^{-1}\ p_\sigma(z_0 - H(x))\ \ \mathrm{d}\mu^-(x)$$

Bayesian filter: assimilation step

Let's do *one* assimilation step at $t = 0$.

- Suppose we are given prior distribution of x_0 μ_0^-.
- Then we can assimilate observation z_0 to get posterior:

$$\mathrm{d}\mu_0^+(x) = \mathbb{P}\left(X_0 = x \mid H(X) + \xi = z_0\right), \; X_0 \sim \mu_0^-$$

- If ξ has pdf p_σ then Bayes' law says

$$\mathrm{d}\mu_0^+(x) = \underbrace{Z_\sigma^{-1}}_{\text{const.}} \underbrace{p_\sigma(z_0 - H(x))}_{C^1 \text{ function}} \underbrace{\mathrm{d}\mu^-(x)}_{\text{old measure}}$$

Bayesian filter: assimilation step

Let's do *one* assimilation step at $t = 0$.

- Suppose we are given prior distribution of x_0 μ_0^-.
- Then we can assimilate observation z_0 to get posterior:

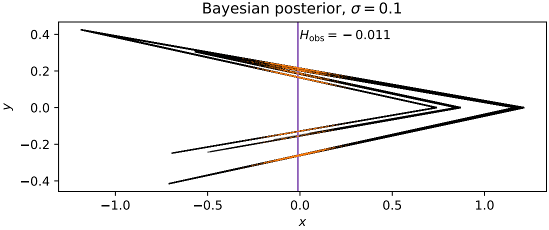
$$\mathrm{d}\mu_0^+(x) = \mathbb{P}\left(X_0 = x \mid H(X) + \xi = z_0\right), \, X_0 \sim \mu_0^-$$

- If ξ has pdf p_σ then Bayes' law says

$$\mathrm{d}\mu_0^+(x) = \underbrace{Z_\sigma^{-1}}_{\text{const.}} \underbrace{p_\sigma(z_0 - H(x))}_{C^1 \text{ function}} \underbrace{\mathrm{d}\mu^-(x)}_{\text{old measure}}$$

How to choose initial posterior μ_0^-? Natural choice is ρ.

Bayesian filter: assimilation step



Bayesian filter: prediction

- Given posterior μ_0^+ we can predict future:

$$\mu_n = T_*^n \mu_0^+$$

- If $\mu_0^- = \rho$ then

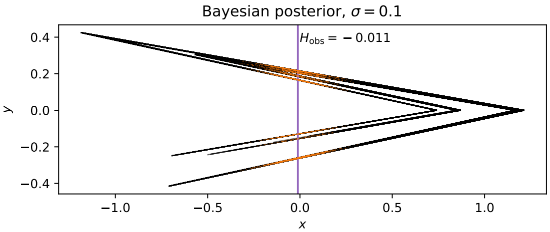
$$\mu_n = T_*^n (Z_\sigma^{-1} p_\sigma(z - H(\cdot))\rho)$$

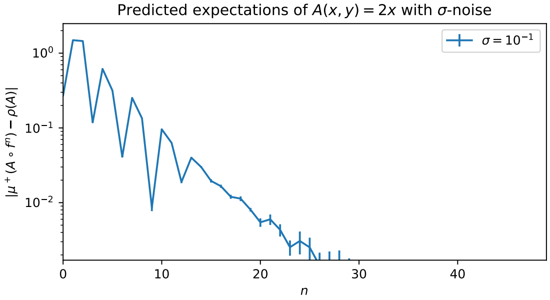
So, as $n \to \infty$,

$$\int A \, \mathrm{d}\mu_n = \int A \circ T^n \; Z_\sigma^{-1} p_\sigma(z - H(\cdot)) \mathrm{d}\rho \to \int A \, \mathrm{d}\rho$$

i.e. our predictions μ_n revert to the “no information" distribution ρ exponentially fast.

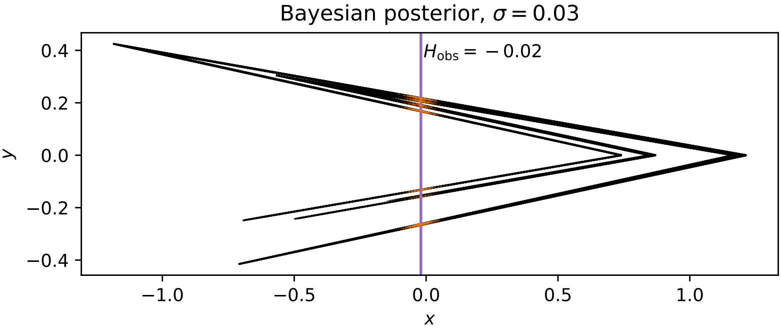
Bayesian filter: prediction



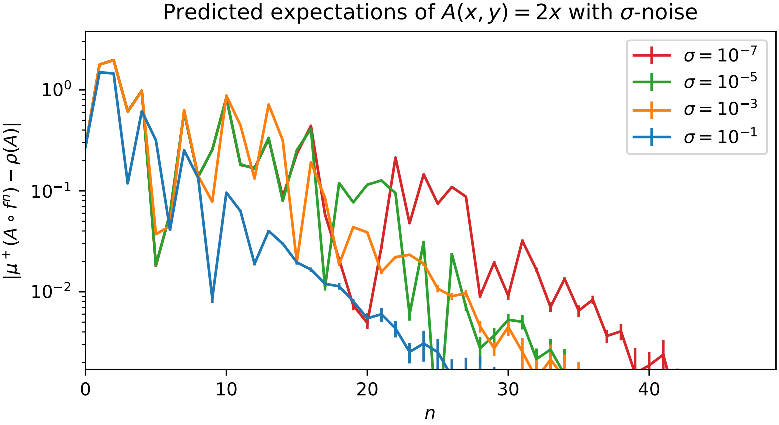


Bayesian filter: prediction

What happens if we improve our observations?



Bayesian filter: prediction



Zero noise limit

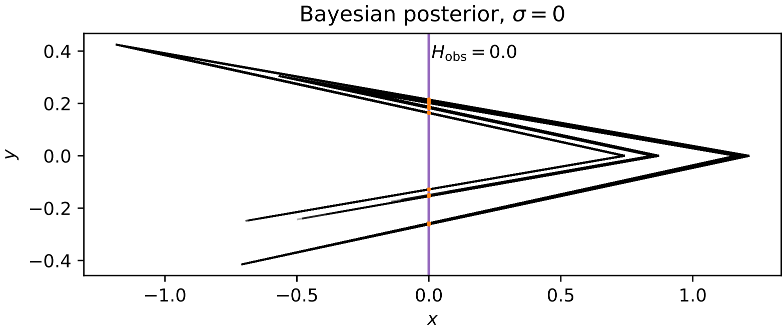
Zero-observation-noise limit?

As $\sigma \to 0$,

$$p_\sigma(z - H(x)) \to \delta(z - H(x))$$

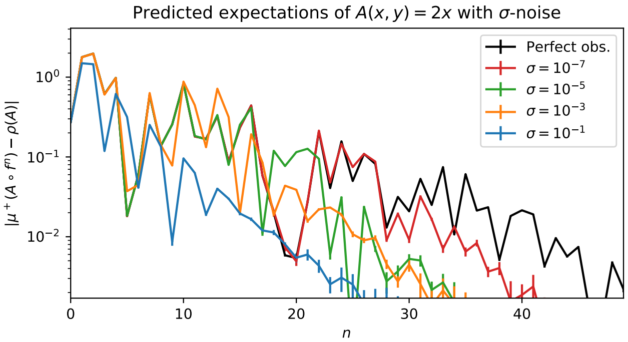
So posterior turns into conditional measure

$$\mathrm{d}\mu_t^+(x) \to Z_0^{-1}\delta(z - H(x))\,\mathrm{d}\rho(x) = \mathrm{d}\rho(x \mid H(x) = z)$$



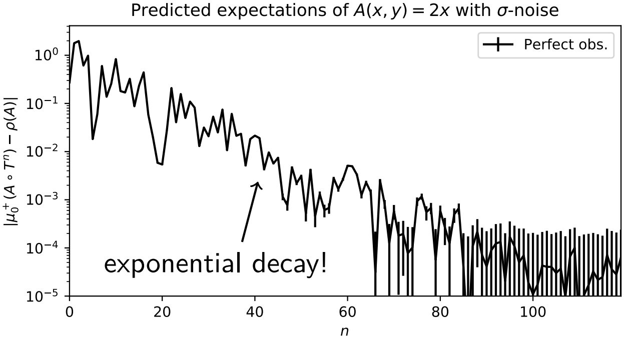
Zero noise limit

With very careful numerics we can make a rigorous sample:



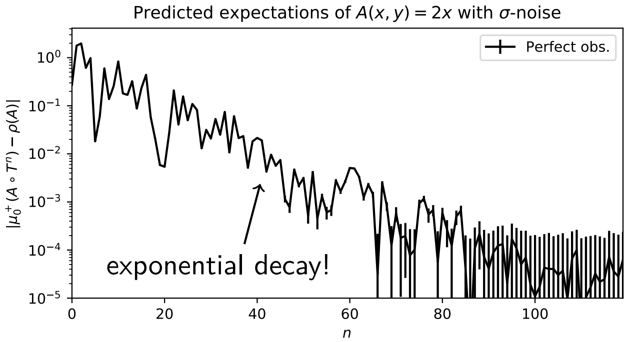
Zero noise limit

With very careful numerics we can make a rigorous sample:



Zero noise limit

With very careful numerics we can make a rigorous sample:



Call this **conditional decay of correlations** (CDoC).

CDoC means that *perfect, partial* observations lose utility over time.

Zero noise limit

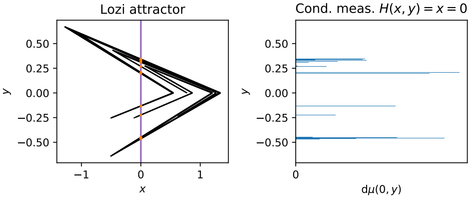
What can we say mathematically?

- If line $z = H(x)$ is a stable manifold or invariant submanifold then $\mu_n^+ \not\rightarrow \rho$.
  - But this is non-generic!
- If system is conservative then CDoC expected
- If system is dissipative, μ^+ is too irregular to prove things (e.g. is Cantor measure)...

Zero noise limit

What can we say mathematically?

- If line $z = H(x)$ is a stable manifold or invariant submanifold then $\mu_n^+ \nrightarrow \rho$.
  - But this is non-generic!
- If system is conservative then CDoC expected
- If system is dissipative, μ^+ is too irregular to prove things (e.g. is Cantor measure)...

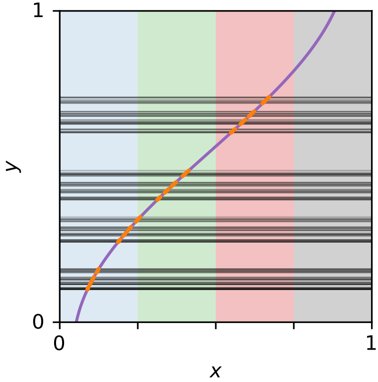


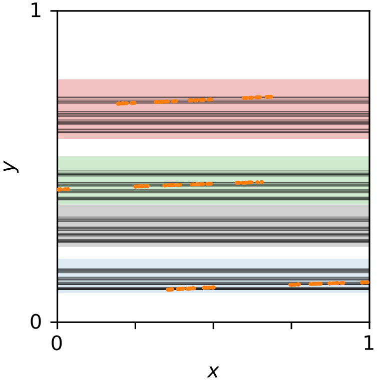
Conditional decay of correlations

Study baker's map on $[0,1]^2$:

$$T(x,y) = (kx \mod 1, v_{\lceil kx \rceil}(y))$$

with v_i non-overlapping contractions.





Conditional decay of correlations

Theorem (W. '22)

The conditional measure $\mu = \rho(\cdot \mid \psi(y) - x = 0)$ *is well-defined. Furthermore, if* $\psi' \neq 0$ *and either*

- *The* $\{v_i\}$ *are analytic and not conjugate to linear.*
- *The* $\{v_i\}$ *are linear with same contraction rate, and* $\psi'' \neq 0$

then there exists $\theta < 1$ *such that*

$$\left| \int A \circ T^n \, \mathrm{d}\mu - \int A \, \mathrm{d}\rho \int \mathrm{d}\mu \right| \leq C\theta^n \|A\|_{C^1}.$$

Super generic: just need to break algebraic structure (linearity)

Conditional decay of correlations

- Conjecture that CDoC occurs if no obvious reason not to
  - i.e. it occurs for almost every observations H, z.
- This gives a *hard* limit on utility of improving observation accuracy.
- CDoC appears to b slowly than regular decay of correlations, maybe dependent on dimension of μ.
- Proving CDoC looks to be hard!

Bayes filter: repeated observations

What if we make $M \geq 1$ observations?

- We can think of it as a single observation at the final time

$$(H, H \circ T^{-n_1}, H \circ T^{-n_2}, \ldots H \circ T^{-n_M}) : \mathbb{R}^d \to \mathbb{R}^{eM}.$$

- If $eM > 2d$ then Takens embedding theorem says we know x exactly $\implies$ exact info for all time!
- Expect CDoC at least if $eM < \#$ positive Lyapunov exponents (can be large for multiscale systems)
- Blender-like properties might imply CDoC for intermediate eM

Linear response (time-independent version)

Given family of maps

$$T^{\varepsilon}(x) = T(x) + \varepsilon X(T(x)) + \dots$$

Linear response asks: when is $\varepsilon \mapsto \rho^{\varepsilon}$ differentiable?
Given formally by sum

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int A \,\mathrm{d}\rho^{\varepsilon} = \sum_{n=0}^{\infty} \kappa_n$$

where κ_n are *susceptibiility coefficients* that may or may not blow up…

Linear response in theory

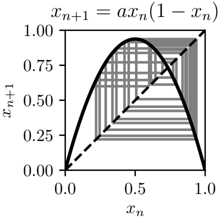
Whe can prove that $\varepsilon \mapsto \rho^\varepsilon$ is:

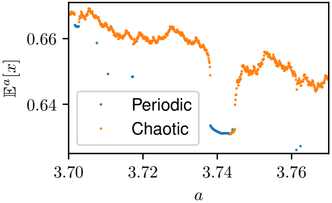
- Differentiable (C^∞) in conservative (Kubo '66) and stochastic (Hairer and Majda '10) systems
- Differentiable (C^∞) in Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97, ...

Linear response in theory

Whe can prove that $\varepsilon \mapsto \rho^{\varepsilon}$ is:

- Differentiable (C^{∞}) in conservative (Kubo '66) and stochastic (Hairer and Majda '10) systems
- Differentiable (C^{∞}) in Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97, ...
- At best $C^{0.5^{-}}$ Hölder in logistic maps (Baladi and others '08-'15)

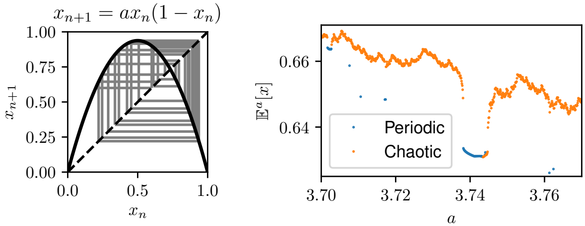




Linear response in theory

Whe can prove that $\varepsilon \mapsto \rho^\varepsilon$ is:

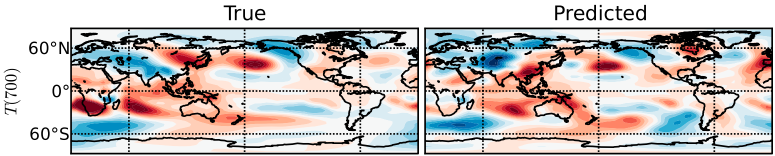
- Differentiable (C^∞) in conservative (Kubo '66) and stochastic (Hairer and Majda '10) systems
- Differentiable (C^∞) in Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97, ...
- At best $C^{0.5^-}$ Hölder in logistic maps (Baladi and others '08-'15)



- Large(r) non-hyperbolic dissipative systems?

Linear response in practice/simulations

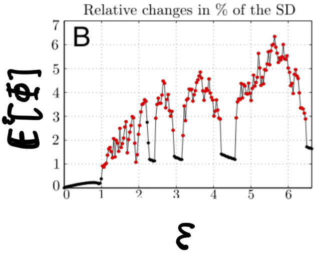
- Long, successful history of applying linear response in climate systems since Leith ('75)



Fuchs et al., 2014

Linear response in practice/simulations

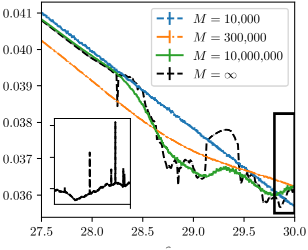
- Long, successful history of applying linear response in climate systems since Leith ('75)
- Some apparent failures (e.g. Cooper and Haynes '13, Chekroun et al. '14)



Chekroun et al., 2014

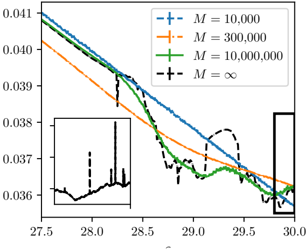
Linear response in practice/simulations

- Long, successful history of applying linear response in climate systems since Leith ('75)
- Some apparent failures (e.g. Cooper and Haynes '13, Chekroun et al. '14)
- Possible to find coupled systems where LR fails in thermodynamic limit (W. and Gottwald '19)



Linear response in practice/simulations

- Long, successful history of applying linear response in climate systems since Leith ('75)
- Some apparent failures (e.g. Cooper and Haynes '13, Chekroun et al. '14)
- Possible to find coupled systems where LR fails in thermodynamic limit (W. and Gottwald '19)



What is the mechanism behind linear response?

Linear response formula

Why?

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int A \,\mathrm{d}\rho_\varepsilon \bigg|_{\varepsilon=0} = -\sum_{n=0}^{\infty} \int A \circ T^n \nabla \quad \cdot (X \mathrm{d}\rho)$$

Linear response formula

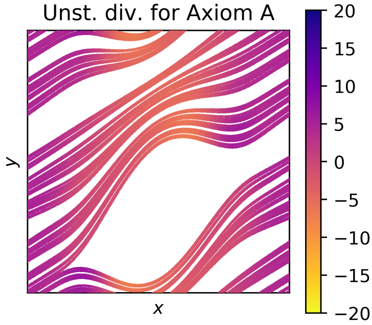
Why?

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int A \, \mathrm{d}\rho_\varepsilon \Big|_{\varepsilon=0} \sim - \sum_{n=0}^{\infty} \int A \circ T^n \nabla^{u} \cdot (X \mathrm{d}\rho)$$

Linear response formula

Why?

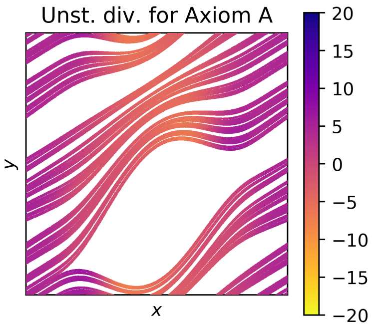
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_\varepsilon\Big|_{\varepsilon=0} \sim -\sum_{n=0}^{\infty}\int A\circ T^n \underbrace{\frac{\mathrm{d}\nabla^u\cdot(X\mathrm{d}\rho)}{\mathrm{d}\rho}}_{\text{Typically a function}}\,\mathrm{d}\rho$$

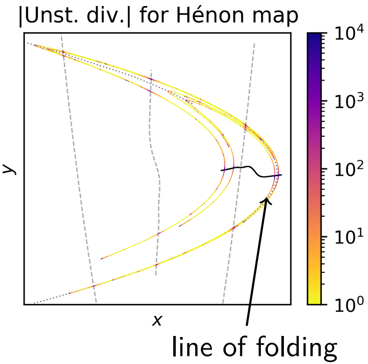


Linear response formula

Why?

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_\varepsilon\Big|_{\varepsilon=0} \sim -\sum_{n=0}^{\infty}\int A\circ T^n \underbrace{\frac{\mathrm{d}\nabla^u\cdot(X\mathrm{d}\rho)}{\mathrm{d}\rho}}_{\text{Typically a function}}\,\mathrm{d}\rho$$

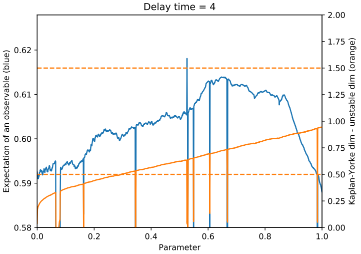




Linear response claim

Ruelle ('11, '18) conjectures:

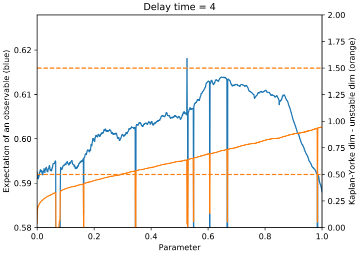
- If you project onto one unstable manifold, singularities in the density even out
- *Stable dimension* d_s large = fatter attractor = more evening out
- Response $\varepsilon \mapsto \rho_\varepsilon$ is $C^{d_s+0.5^-}$.



Linear response claim

Ruelle ('11, '18) conjectures:

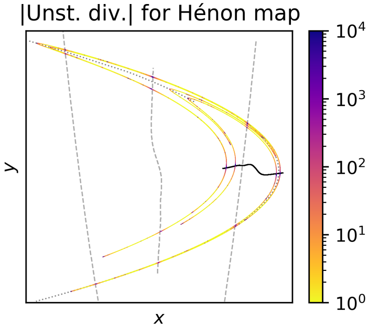
- If you project onto one unstable manifold, singularities in the density even out
- *Stable dimension* d_s large = fatter attractor = more evening out
- Response $\varepsilon \mapsto \rho_\varepsilon$ is $C^{d_s+0.5^-}$.



How to prove this argument? **No clue, immensely difficult.**

Lozi map

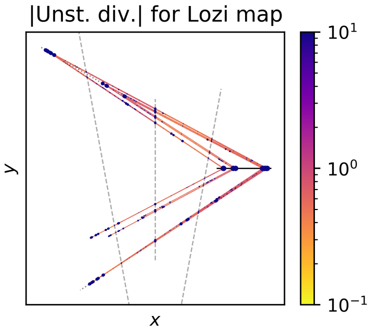
Lozi map simplifies Hénon map:



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ by \end{pmatrix}$$

Lozi map

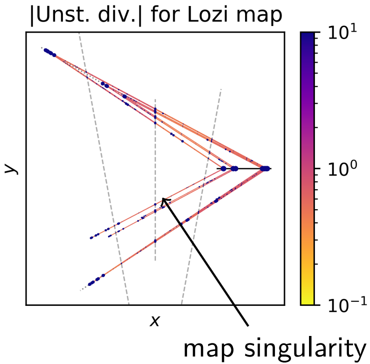
Lozi map simplifies Hénon map:



$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1-a|x|+y\\by\end{pmatrix}$$

Lozi map

Lozi map simplifies Hénon map:



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - a|x| + y \\ by \end{pmatrix}$$

Linear response

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int A \,\mathrm{d}\rho_\varepsilon \bigg|_{\varepsilon=0} = -\sum_{n=0}^{\infty} \int A \circ T^n \underbrace{\nabla^u \cdot (X \mathrm{d}\rho)}_{\neq B\rho}$$

For the Lozi map,

$$\nabla^u \cdot (X \mathrm{d}\rho) = \phi_{\mathrm{cts}} \,\mathrm{d}\rho + \sum_{m=-\infty}^{\infty} \phi_m \,\mathrm{d}T_*^m \mu$$

where $\mu = \rho(\cdot \mid \text{singularity line})$

Linear response

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int A\,\mathrm{d}\rho_\varepsilon\bigg|_{\varepsilon=0} \sim -\sum_{n=0}^{\infty}\int A\circ T^n\,\underbrace{\nabla^u\cdot(X\mathrm{d}\rho)}_{\neq B\rho}$$

For the Lozi map,

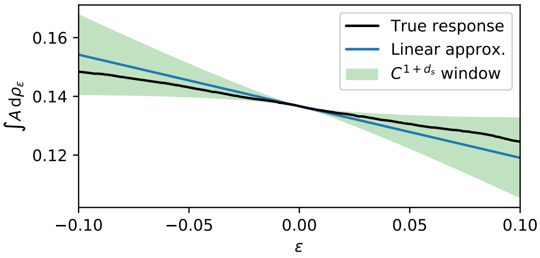
$$\nabla^u\cdot(X\mathrm{d}\rho) = \underbrace{\phi_{\mathrm{cts}}}_{\text{function}}\,\mathrm{d}\rho + \sum_{m=-\infty}^{\infty}\underbrace{\phi_m}_{\text{functions}}\,\mathrm{d}T_*^m\mu$$

where $\mu = \rho(\cdot \mid \text{singularity line})$

Linear response

Theorem (W. '22)

Conditional decay of correlations (slightly extended) for the Lozi map implies that the linear response formula converges.



Linear response

How can this help us with smooth systems (Hénon, climate models. . .)?

- Singularities again located on orbit of singularity “lines" (now stable/unstable tangencies)
- If we have conditional decay of correlations on these “lines" in largish systems, then linear response is likely to hold
- BUT: we expect susceptibility functions to decay slower than usual rate of mixing

Conclusion

- "Conditional decay of correlations" regulates
  - Linear response in non-hyperbolic dissipative chaos (i.e. complex systems)
  - Long-term prediction from accurate observations
- Mathematically CDoC is new ground
- Practical implications? Better LRT algorithms?

Hot off presses: `arXiv:2206.09291`
`arXiv:2206.09292`