Emergence and breakdown of linear response in globally coupled systems

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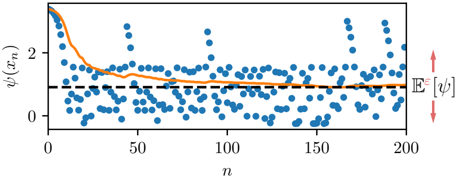
Joint work with Georg Gottwald

Setting

Consider family of chaotic systems $x_n = T^\varepsilon(x_{n-1})$, with physical measures ρ^ε.
The physical measures encode the long-term ergodic behaviour for each T^ε. For observables ψ and Lebesgue-a.e. x_0,

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi(x_n) \xrightarrow{N\to\infty} \int \psi(x)\,\mathrm{d}\rho^\varepsilon(x) =: \mathbb{E}^\varepsilon[\psi]$$



Linear response theory

$$\mathbb{E}^{\varepsilon}[\psi] := \int \psi(x)\,\mathrm{d}\rho^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}^{\varepsilon}[\psi]$*?*
(e.g. for Taylor approximations)

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…supposing $\varepsilon \mapsto \mathbb{E}^{\varepsilon}[\psi]$ is differentiable

LRT in theory

Analytically, we know LRT works in

- Statistical mechanics: Kubo '66
- Stochastic dynamical systems: Hänggi '78, Hairer & Majda '10
- Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97-8
- Other dissipative systems. . . ?

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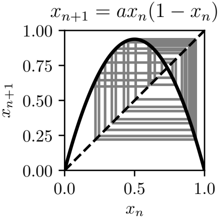
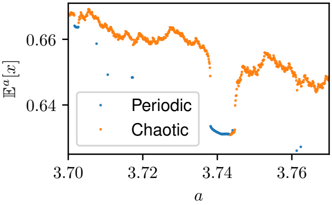
Baladi and others ('08, '10, '14, '15) proved there is no **linear response for quadratic maps, even Whitney differentiability**.

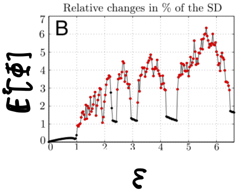




LRT in practice

Geophysicists have applied LRT to climate systems:

- A long record of success!
- Justified by *chaotic hypothesis*: "macroscopic dynamics are Axiom A"
- However, linear response appears to fail in some systems (e.g. Chekroun *et al.* '14, Cooper and Haynes '13)



Chekroun *et al.*, '14

The question

We address the following question:

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We study a "simple complex system" of M chaotic maps coupled via a mean-field at

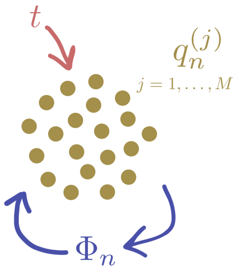
I. Infinite size M (thermodynamic limit)

II. M finite but fairly large

Globally coupled maps

$$q_n^{(j)} = f_{\varepsilon,\Phi_{n-1}}(q_{n-1}^{(j)}), \; j = 1, \ldots, M$$

$$\Phi_n = \frac{1}{M}\sum_{j=1}^{M} \phi(q_n^{(j)})$$



Observable is mean field Φ_n.

These systems have rich dynamical and linear response behaviour.

Model reduction

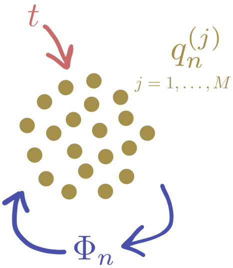
Use exchangeability of subsystems $q^{(j)}$ to write in terms of empirical measure:

$$\mu_n = \frac{1}{M} \sum_{j=1}^{M} \delta_{q_n^{(j)}}.$$

System becomes

$$\mu_n = f^*_{\varepsilon,\Phi_n} \mu_{n-1}$$

$$\Phi_n = \int \phi \, \mathrm{d}\mu_n$$



Model reduction: thermodynamic limit

In thermodynamic limit ($M \to \infty$) expect μ_n to be a physical measure of cocycle $\{f_{\varepsilon,\Phi_n}\}_{n\in\mathbb{N}}$:

$$\mu_n = \mu_n(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) := \lim_{k\to\infty} f^*_{\Phi_{n-1},\varepsilon} \cdots f^*_{\Phi_{n-k},\varepsilon} \mathrm{Leb}$$

This gives us delay system in Φ:

$$\Phi_n = \int \phi \, \mathrm{d}\mu_n^\infty =: F_\varepsilon(\Phi_{n-1}, \Phi_{n-2}, \ldots)$$

What are its dynamics?

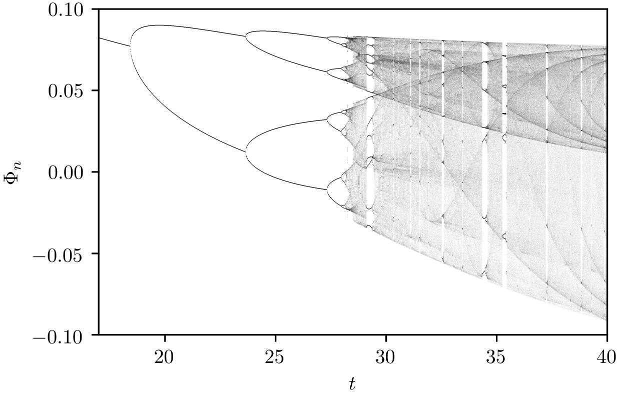
Model reduction: thermodynamic limit

- Mixing of microscopic dynamics f implies F only depends on recent history of Φ.
  Macroscopic dynamics are close to finite dimensional!
- For any map g we can find a coupled system with $\Phi_n \approx g(\Phi_{n-1})$.
  All dynamics are possible.

For F to be smooth, we need f to have linear response.

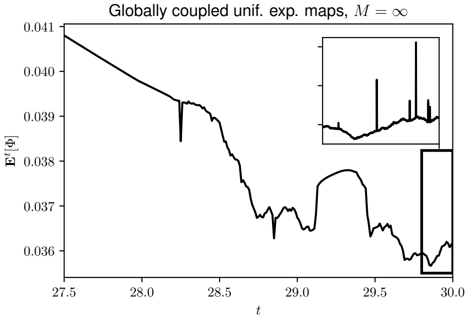
Thermodynamic limit

We studied f uniformly expanding (**best possible case** for LRT).
We can obtain a period doubling bifurcation to (macroscopic) chaos:



Thermodynamic limit

Despite hyperbolic subsystems, a **failure of linear response**:



In fact, the macroscopic dynamics are **non-hyperbolic**, contra Gallavotti-Cohen chaotic hypothesis.
Evidence: homoclinic tangencies.

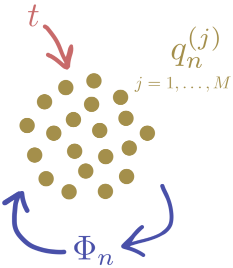
Finite size

In climate science have incomplete scale separations: need to consider **finite size effects**.

Mean-field Φ_n is more or less a random sample from the thermodynamic limit.
So by **central limit theorem**,

$$\Phi_n = F_\varepsilon(\Phi_{n-1}, \Phi_{n-2}, \ldots) + \frac{1}{\sqrt{M}}\zeta_n,$$

where ζ_n is a mean-zero Gaussian process with decay of correlations etc.

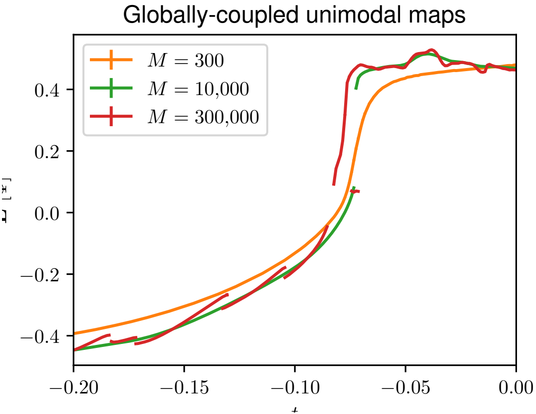


This is a **stochastic system.**

$\implies$ we expect LRT

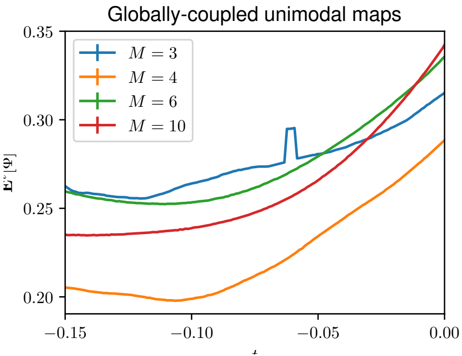
Finite size

Noise induces linear response even when f is nasty (e.g. quadratic map):



Finite size

Works even for very small systems (e.g. $M = 4$):



Likely related to generic distribution of singularities in physical measures (Ruelle '19, W. in progress).

Conclusions

Studied globally coupled systems via models for macroscopic dynamics for large/infinite M.

- Finite size: emergent stochastic effects reliably induce linear response
- In thermodynamic limit need:
  - Microscopic dynamics satisfy LRT
  - Macroscopic dynamics are nice, e.g. hyperbolic (**not** always true)
- Not shown: parameter variation in subsystems helps produce mean-field LRT
- Q: how does this extend to other kinds of couplings?

Further details

Wormell, C.L. and Gottwald, G.A., 2019. Linear response for macroscopic observables in high-dimensional systems. arXiv:1907.13490.

Wormell, C.L., in preparation. Homoclinic tangencies in the macroscopic dynamics of a globally coupled system.