Emergence of linear response at macroscopic scales

Caroline Wormell and Georg Gottwald

Introduction

Consider a smooth family of **chaotic systems** $x_n = G(x_{n-1}, \varepsilon)$, with **physical measures** μ^ε encoding the system's equilibrium statistics.

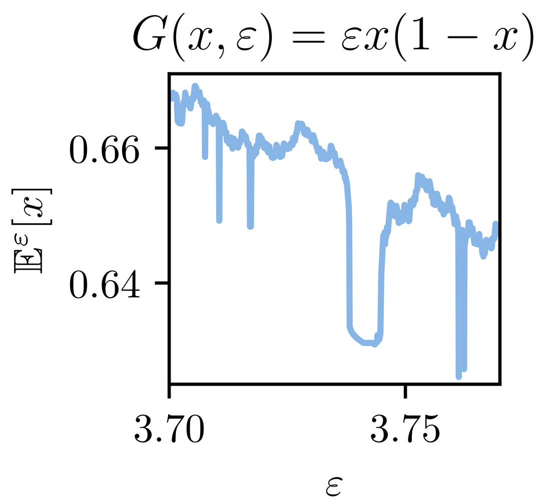
A system has **linear response** if for "nice" observables Φ,

$$\mathbb{E}^\varepsilon[\Phi] := \int \Phi \, d\mu^\varepsilon$$

is differentiable with respect to ε. Derivatives can be calculated using **linear response theory (LRT)**.

LRT has been **successfully applied** to various climate models (e.g. Lucarini *et al.* '10).

However, some systems exhibit a **failure of LRT** (e.g. Chekroun *et al.* '14), and recently Baladi and others proved that **logistic maps have no linear response** (see right).



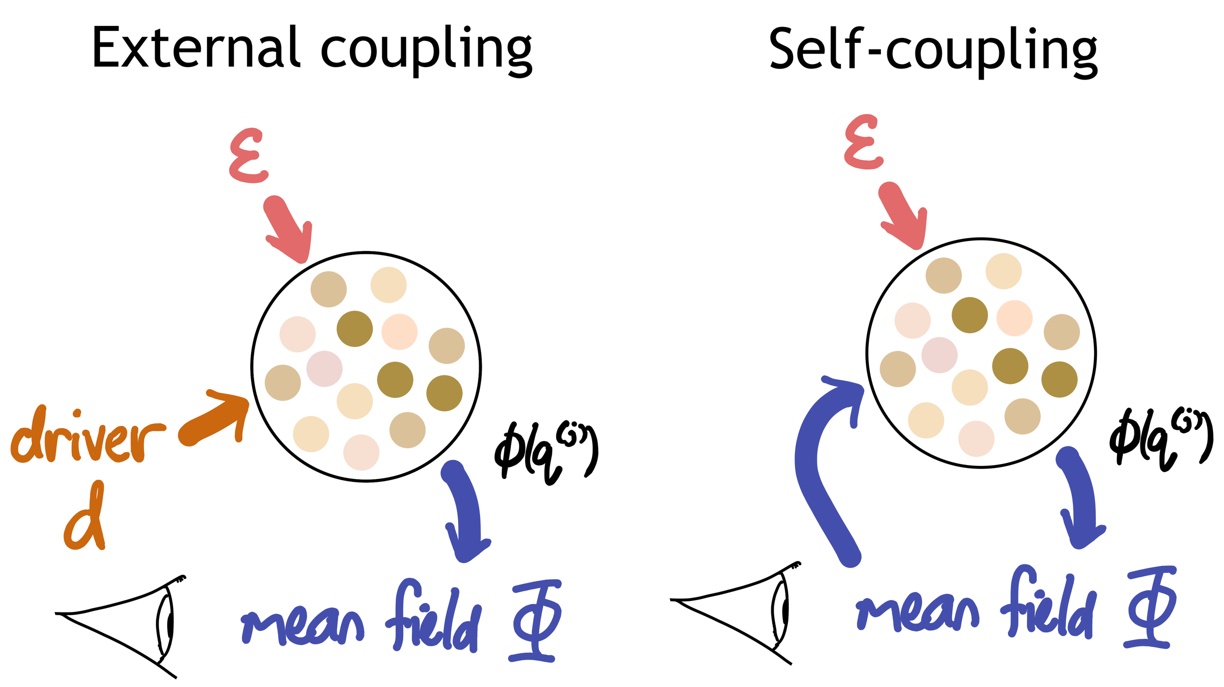
What determines the existence of linear response in complex systems at large spatial scales?

Model

We consider an **inhomogeneous ensemble** of $M \gg 1$ chaotic subsystems $q^{(j)}$, with **random system parameters** $a^{(j)} \sim \nu$.

We **observe** a **mean field** Φ.

We consider **two different couplings**:



Dynamics:

$$\begin{cases} q_n^{(j)} &= f(q_{n-1}^{(j)}; d_{n-1}/\Phi_{n-1}, a^{(j)}, \varepsilon) \\ \Phi_n &= \frac{1}{M}\sum_{j=1}^{M} \phi(q_n^{(j)}) \end{cases}$$

Macroscopic reduction

All parameters derived from **averaged microscopic statistics**

- **Externally coupled system** has large-M **CLT reduction**

$$(1) \qquad \Phi_n = F(d_{n-1}, d_{n-2}, \ldots; \varepsilon) + M^{-1/2}\eta_n^{d,\varepsilon} + M^{-1/2}\zeta_n^{d} + o(M^{-1/2})$$

$$:= \iint \phi(q)\, d\mu_n^{d,a,\varepsilon}(q)\, d\nu(a)$$

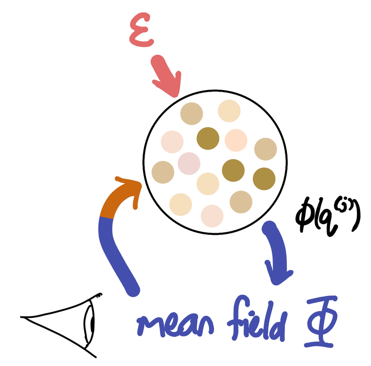
where $\mu_n^{d,a,\varepsilon}$ are the time-dependent physical measures of cocycle $q_{n+1} = f(q_n; d_n, a, \varepsilon)$

Variation in selection of parameters $a^{(j)}$ (i.e. pre-set for dynamics)

Dynamical variation **Centred Gaussian process** with **decay of correlations**

- For **self-coupled system**, coupling can be modelled as **external**:

$$(2) \qquad \Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) + M^{-1/2}\eta_n^{\Phi,\varepsilon} + M^{-1/2}\zeta_n^{\Phi,\varepsilon} + o(M^{-1/2})$$



Microscopic inhomogeneity

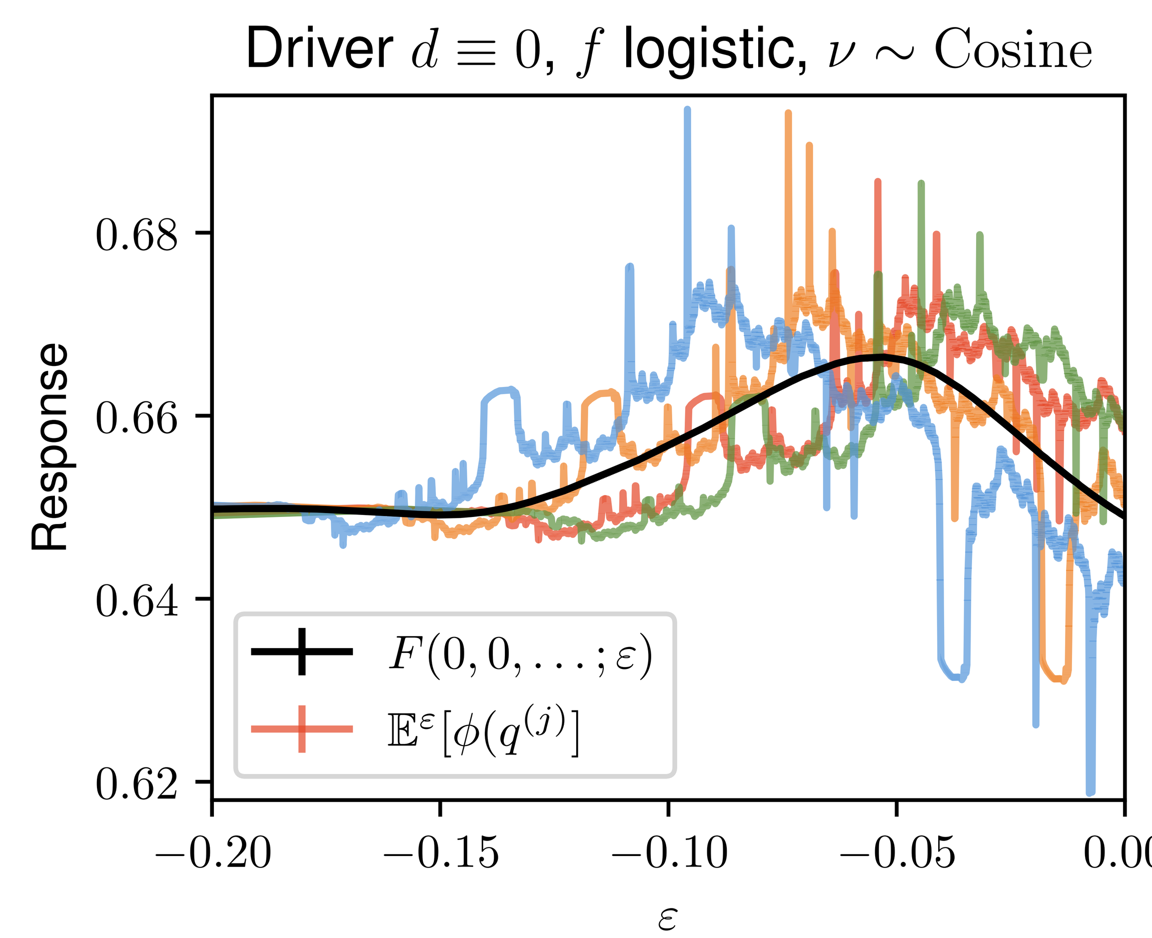
Consider the externally coupled system. From equation (1),

$$\mathbb{E}^\varepsilon[\Phi_n] = \iint \phi(q)\, d\mu^{d,a,\varepsilon}\, d\nu(a) + O(M^{-1/2})$$

So response of Φ is, to first order in M, the **average response** of the **microscopic subsystems**.

Suppose the microscopic variables do not have linear response. If the parameter distribution ν is discrete, neither does Φ, but however **if the microscopic variables are appropriately heterogeneous (ν smooth), this may induce linear response in the macroscopic variables**:

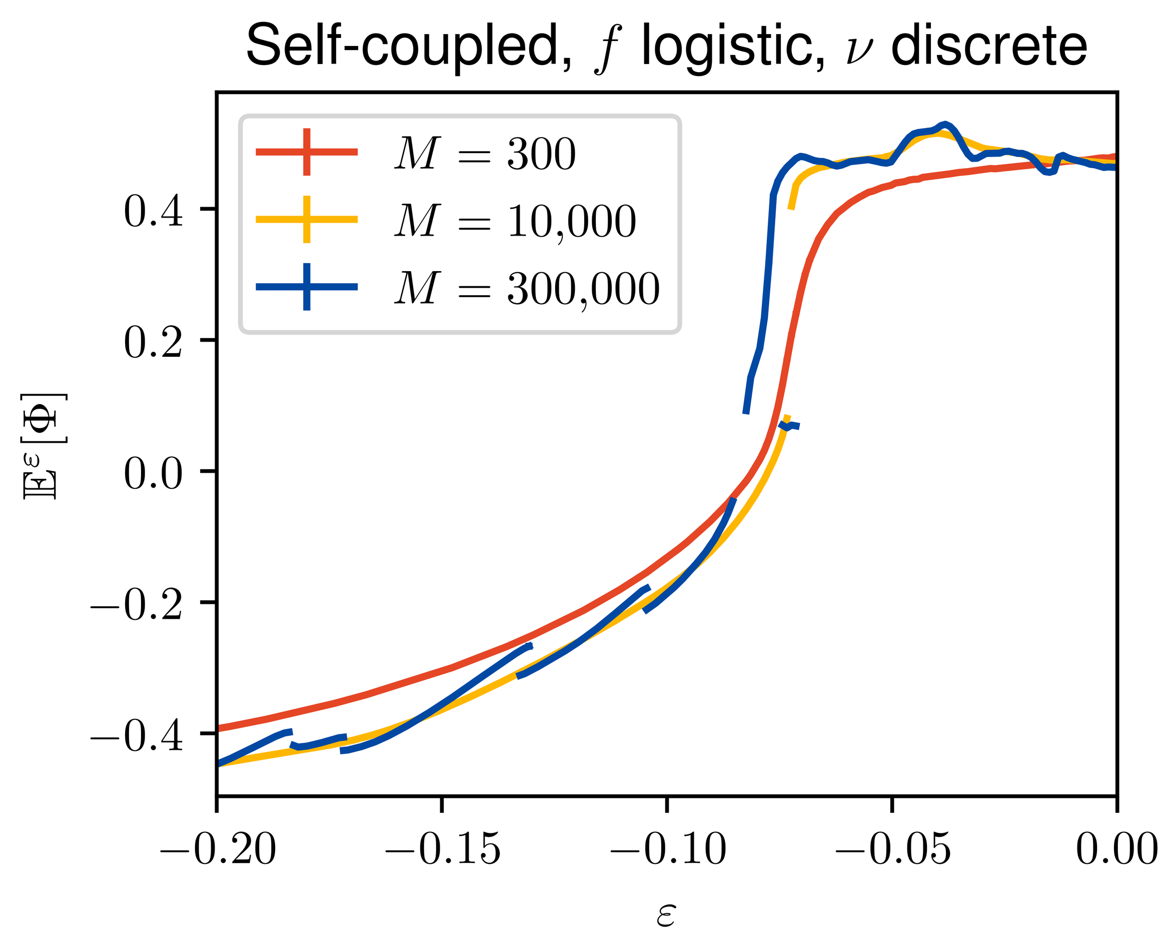
- If $f(\cdot\,; d, a, \epsilon) = f(\cdot\,; d, \epsilon + Ka)$ then mean field Φ has linear response **irrespective of** f.
- If f is **unimodal** (e.g. logistic) then via a conjecture of Avila *et al.*, for at least $d \equiv 0$ the mean field Φ **has linear response** (below)



Self-coupled finite ensemble

Consider a **self-coupled system** with $M < \infty$. Equation (2) defines a **stochastic dynamical system** via the ζ_n process. It is appropriately **smooth** if the microscopic variables obey LRT (which ζ_n induces). Thus, **self-coupled finite ensembles have linear response**, **independent of the microscopic variables.**

This may break down as $M \to \infty$, e.g. via bifurcations:



Mean-field chaos

In the **thermodynamic limit** ($M \to \infty$) for the **self-coupled** system, equation (2) becomes

$$\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon)$$

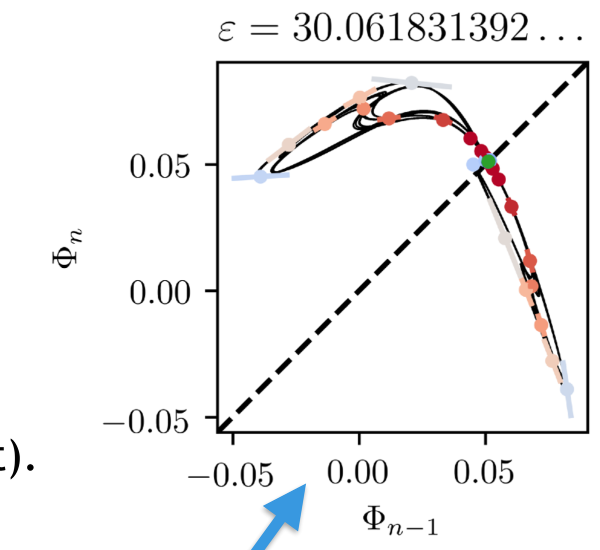
Microscopic mixing means we can ignore Φ_{n-l} for l large.

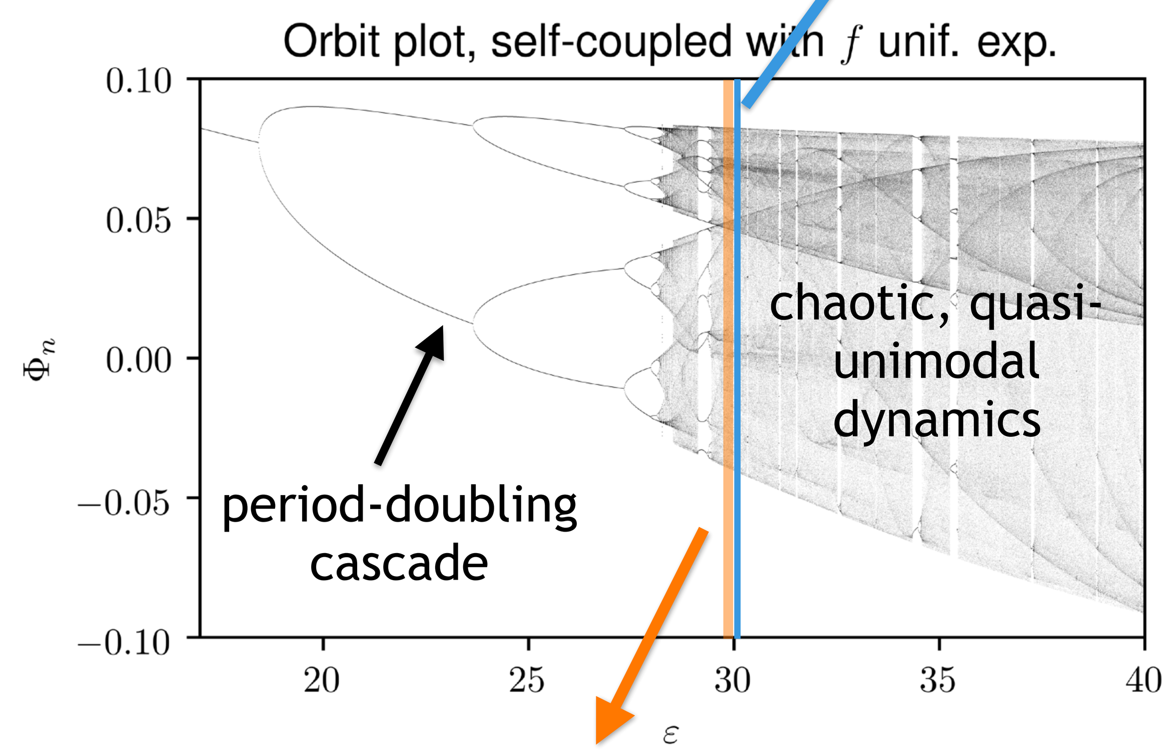
If F is smooth this is a **deterministic** smooth low-dimensional dynamical system.

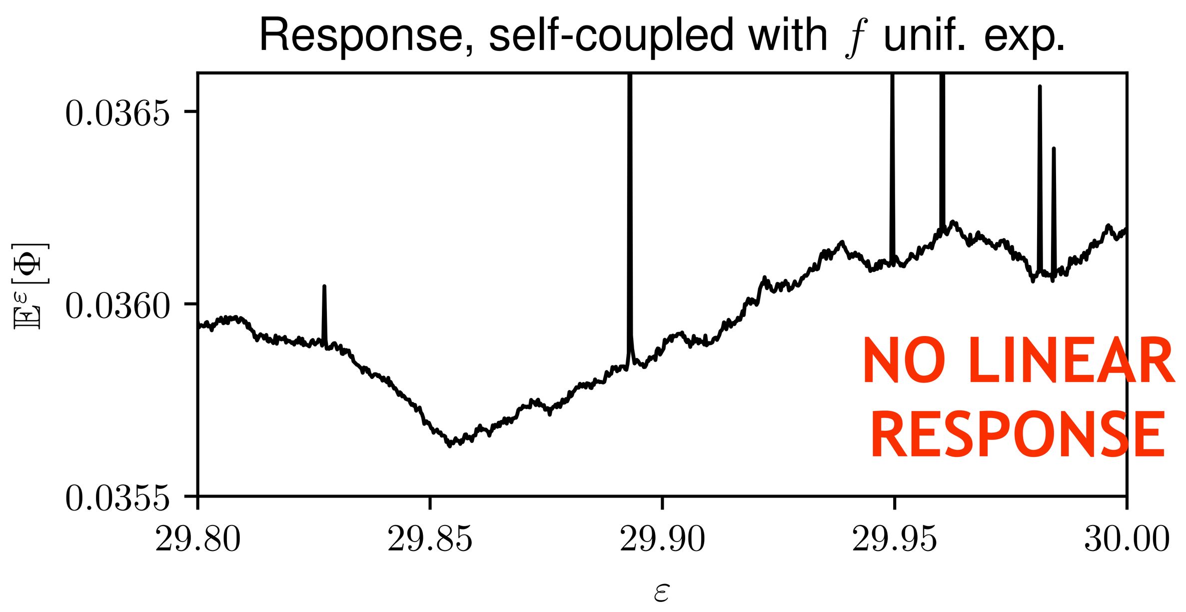
$\Longrightarrow$ **low-dimensional bifurcation theory works**

We used Chebyshev basis transfer operator methods in Poltergeist.jl to *accurately* (13 d.p.) simulate F dynamics for f uniformly expanding.

We find **LRT-violating, non-hyperbolic chaos in macroscopic dynamics**, despite **hyperbolic microscopic variables.** We confirmed non-hyperbolicity numerically by finding **homoclinic tangencies** (right).







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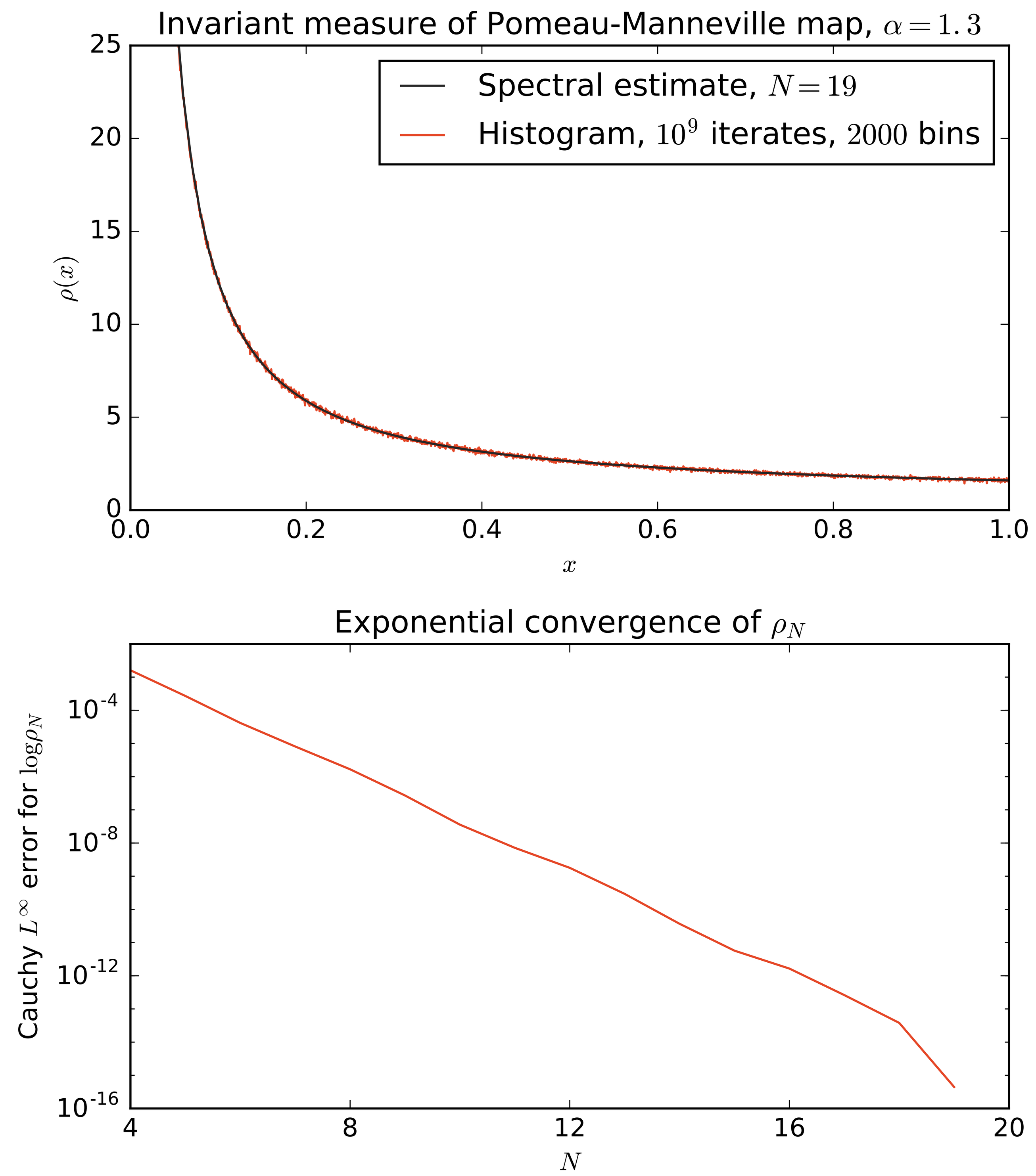
Invariant measure of Pomeau-Manneville map, $\alpha = 1.3$
Spectral estimate, $N = 19$
Histogram, 10^9 iterates, 2000 bins
$\rho(x)$
x
Exponential convergence of ρ_N
Cauchy L^∞ error for $\log \rho_N$
N

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Introduction

Consider a smooth family of **chaotic systems** $x_n = G(x_{n-1}, \varepsilon)$, with **physical measures** μ^ε encoding the system's equilibrium statistics.

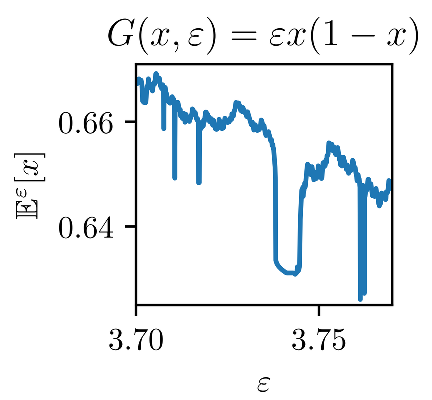
The system has a <u>**linear response**</u> if for "nice" observables Ψ,

$$\mathbb{E}^\varepsilon[\Psi] := \int \Psi \, \mathrm{d}\mu^\varepsilon$$

is differentiable with respect to ε. Derivatives can be calculated using **linear response theory (LRT)**.

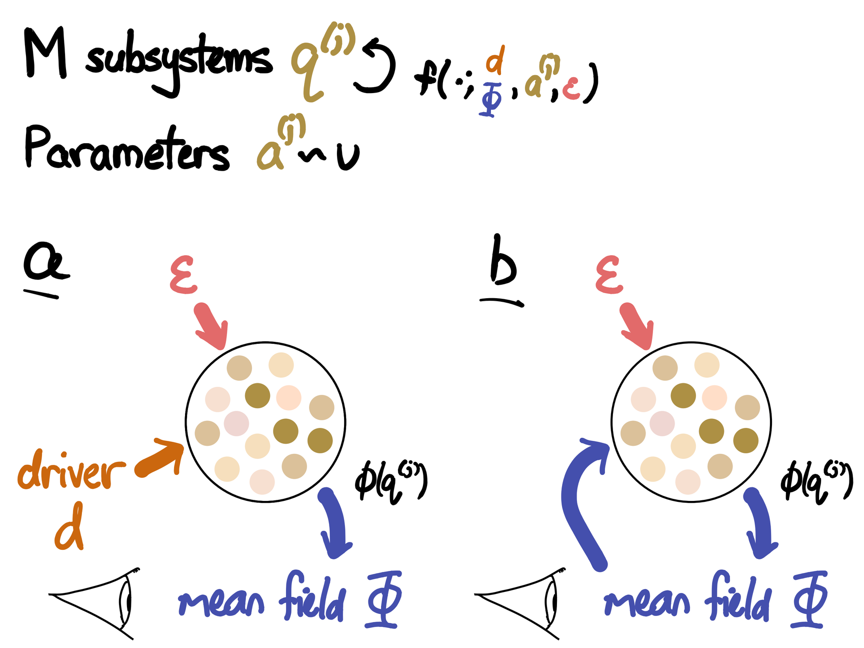
LRT has been applied to climate models with **some success**, ??particularly at **macroscopic scales** (cite survey?).

However, some systems exhibit a **failure of LRT** (e.g. Chekroun *et al.* '14), and recently Baladi and others proved that **logistic maps have no linear response** (see right).



What determines the existence of linear response at large scales in complex systems?

Two "simple" complex systems



Macroscopic reduction

Uncoupled system has reduction

$$\Phi_n = F(d_{n-1}, d_{n-2}, \ldots; \varepsilon) + M^{-1/2}(\zeta_n + \eta_n^{d,\varepsilon}) + o(M^{-1/2}), \quad (1)$$

where

$$F(d_{n-1}, d_{n-2}, \ldots; \varepsilon) := \iint \phi \, \mathrm{d}\mu_n^{d,a,\varepsilon} \, \mathrm{d}\nu(a)$$

and $\mu_n^{d,a,\varepsilon}$ are the time-dependent physical measures of the cocycle $q_{n+1} = f(q_n; d_n, a, \varepsilon)$.

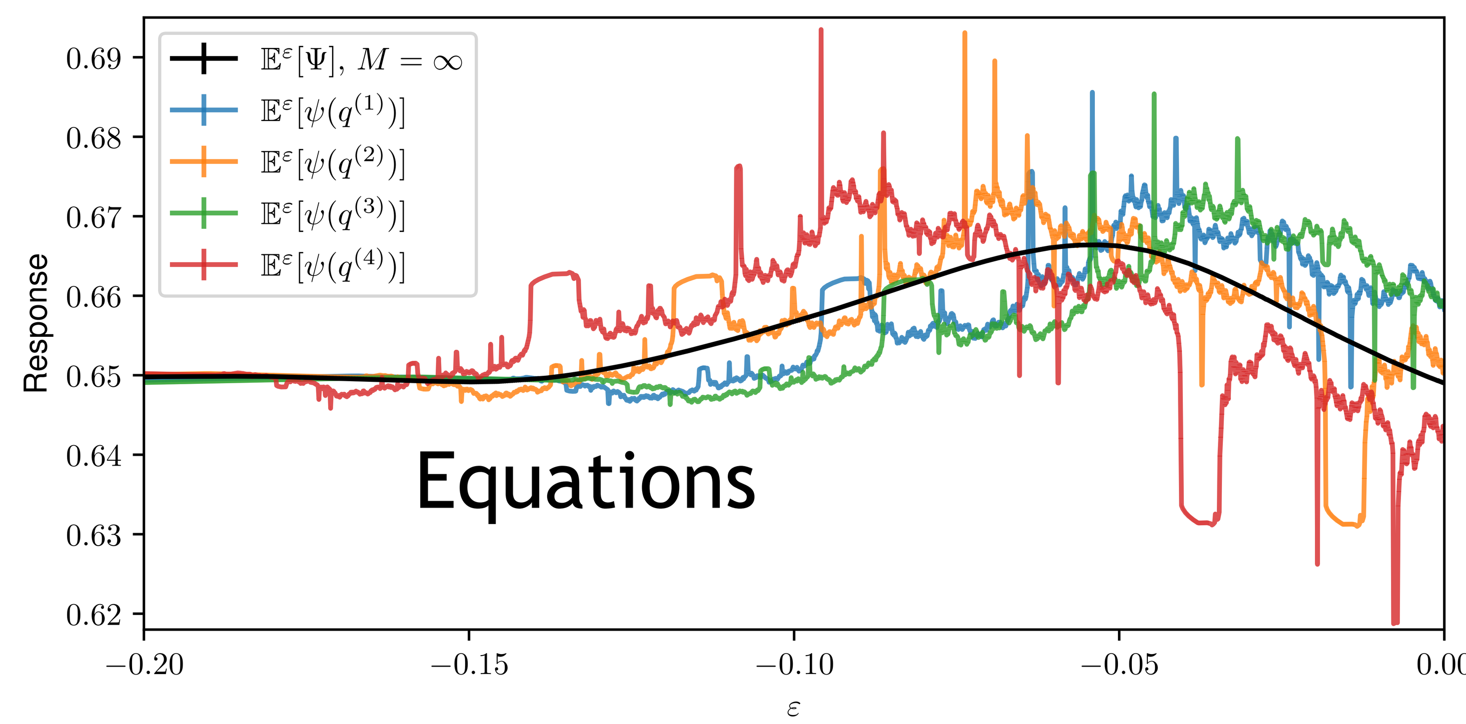
Path 1: "collective" LRT

Consider the first model and for simplicity that $d_n \equiv 0$. From (1),

$$\mathbb{E}^\varepsilon[\Phi] = \iint \phi(q) \, \mathrm{d}\mu^{0,a,\varepsilon} \, \mathrm{d}\nu(a) + O(M^{-1/2}).$$

So the response of Φ is, to first order, the **average response** of the **microscopic subsystems**.

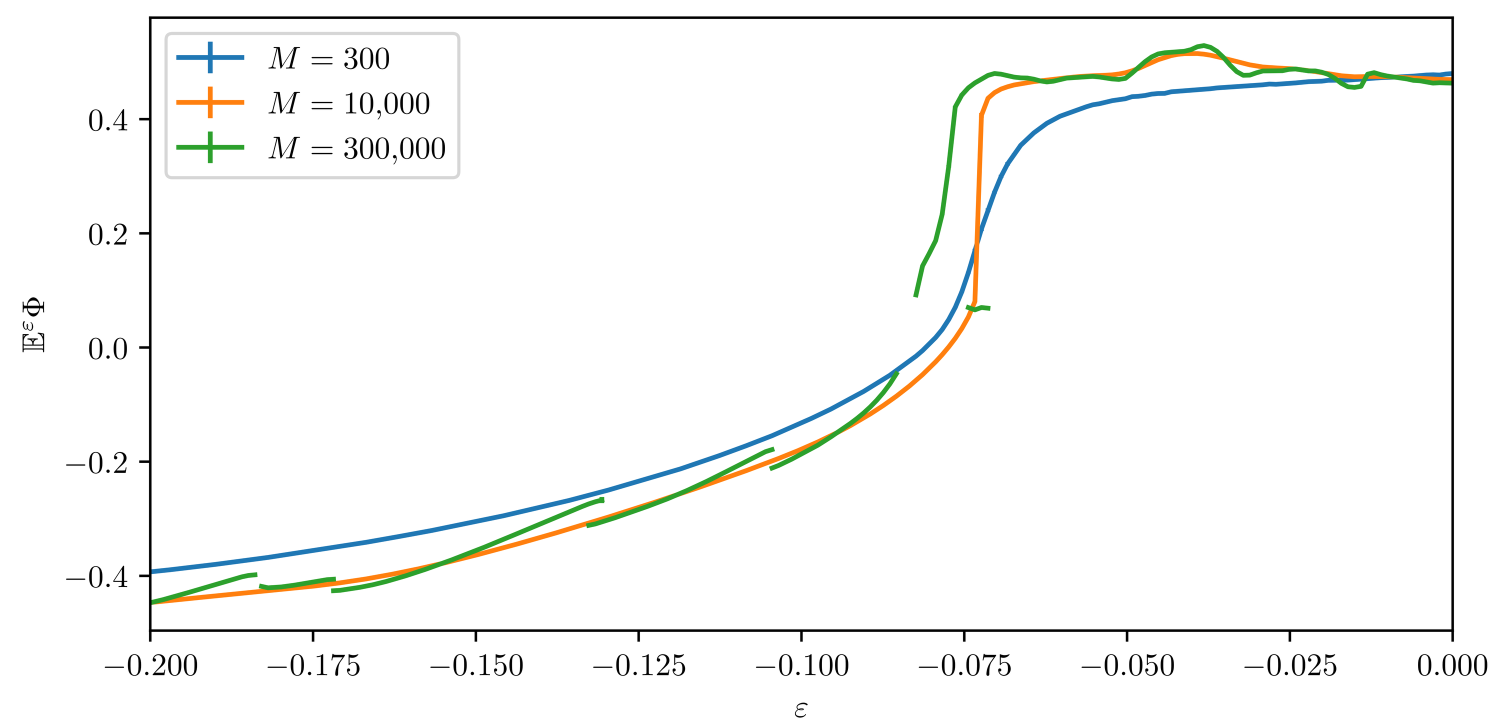
- If the microscopic subsystems have LR then so does Φ; but,
- If $f(\cdot\,; d, a, \epsilon) = f(\cdot\,; d, \epsilon + Ka)$ then Φ has LR **irrespective of f**.
- If f is unimodal (e.g. **logistic**) then via a conjecture of Avila and others, the mean field Φ **has linear response** (see figure below)



Path 2: emergent noise

For $M < \infty$, equation (2) defines a **stochastic dynamical system** due to the ζ_n process. It is appropriately **smooth** if the microscopic variables obey LRT (which ζ_n inducess). Thus, it has linear response.

This may break down as $M \to \infty$, e.g. by dense bifurcations:



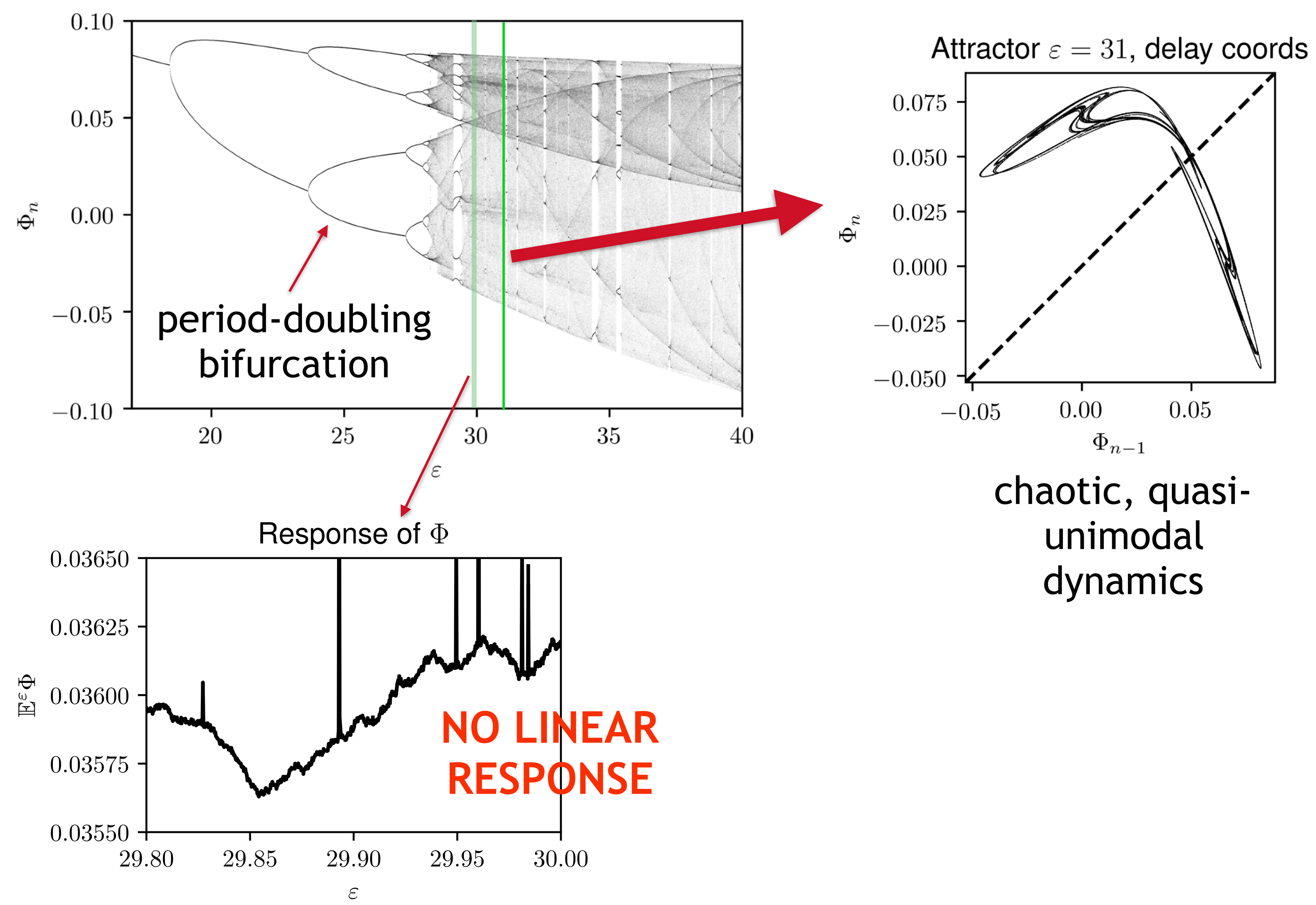
Anti-Path: thermodynamic limit

In the self-coupled case, as $M \to \infty$ we have deterministic dynamics:

$$\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon).$$

Microscopic mixing means we can ignore Φ_{n-l} for l large. With collective LRT this is a smooth **low-dimensional dynamical system**.

With `Poltergeist.jl` we used transfer operator methods to *very* accurately simulate the dynamics of F for **uniformly hyperbolic microscopic** variables. Find **non-hyperbolic chaos** (confirmed by numerically finding **homoclinic tangencies**).



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I'm giving a talk on this here, 11am Tuesday